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A LOCUS IN [8] INVARIANT UNDER A GROUP OF ORDER 51840×81

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1. Clifford matrices and groups for [8]

In [8], the complex projective space of eight dimensions, a point is specified here by an ordered set of nine homogeneous coordinates $x_{00}, x_{10}, x_{20}, x_{01}, x_{11}, x_{21}, x_{02}, x_{12}, x_{22}$. Vertices of the simplex of reference are $A_{00}, A_{10}, A_{20},..., A_{22}$ opposite the respective prime faces $x_{00} = 0, x_{10} = 0, x_{20} = 0,..., x_{22} = 0$.

A *Clifford matrix* $W^{aa'}_{bb'}$ is a 9-by-9 matrix having one non-zero element in each row and column, namely, $1, \epsilon$, or ϵ^2, where $\epsilon = \exp \frac{2}{3}\pi i$. Subscripts and superscripts assume the values 0, 1, 2, and all arithmetical functions of these indices are reduced modulo 3. Consequently, there are $3^4 = 81$ Clifford matrices of size 9. Number the rows and columns in the reversed ternary scale: that is, in the manner of the suffixes of the coordinates. Suppose the elements of $W^{aa'}_{bb'}$ to be $\alpha^{aa'}_{bb'}$, superscripts referring to rows and subscripts to columns. Then, by Room (**1**), the non-zero element in each row and column is given by

$$\alpha^{c\ \ c'}_{c+b\ c'+b'} = \epsilon^{ac+a'c'} \quad (c, c' = 0, 1, 2).$$

Key properties of the Clifford matrices are

$$(W^{aa'}_{bb'})^3 = I \ (= W^{00}_{00}), \tag{i}$$

$$W^{aa'}_{bb'}\, W^{cc'}_{dd'} = \epsilon^{bc+b'c'} W^{a+c\ a'+c'}_{b+d\ b'+d'}, \tag{ii}$$

$$W^{cc'}_{dd'}\, W^{aa'}_{bb'} = \epsilon^{ad+a'd'-bc-b'c'} W^{aa'}_{bb'}\, W^{cc'}_{dd'}. \tag{iii}$$

A *Clifford set* is a set of Clifford matrices (say W_1, W_2, W_3, W_4, W_5) such that

$$W_s W_r = \epsilon W_r W_s \quad (r < s;\ r, s = 1, 2, 3, 4, 5). \tag{iv}$$

Morinaga and Nono (**2**) have shown that such a set contains a maximum of 5 members, and that, for the fifth member,

$$W_5 = (W_1)^2 W_2 (W_3)^2 W_4,$$

except for a possible scalar factor (a power of ϵ).

As *basic Clifford set*, choose

$$W^{10}_{00},\ W^{10}_{10},\ W^{01}_{10},\ W^{01}_{11},\ W^{00}_{11}.$$

The 81 Clifford matrices $W^{aa'}_{bb'}$ form the *Clifford collineation group CG*.

From CG is derived the *Clifford substitution group* CS which is the group of those operations transforming the basic Clifford set into other Clifford sets. By (**3**), the order of CS is 51840. This number is the order of the group of the automorphisms of the lines of the cubic surface in ordinary projective space. A subgroup of this group, of index 2, plays a prominent role later.

Again, by (**3**), any five consecutive matrices of the *Clifford cycle*

$$W^{10}_{00},\ W^{10}_{10},\ W^{01}_{10},\ W^{01}_{11},\ W^{00}_{11},\ W^{20}_{00},\ W^{20}_{20},\ W^{02}_{20},\ W^{02}_{22},\ W^{00}_{22},\ W^{10}_{00},\ldots$$

constitute a *Clifford set.*

The indices of $W^{aa'}_{bb'}$ may be represented by the *index vector* $\begin{bmatrix} a \\ b \\ a' \\ b' \end{bmatrix}$.

Indices of the 4 independent matrices of a Clifford set, arranged in order as the columns of a matrix, form an *index matrix* of size 4. It is proved in (**3**) that CS is isomorphic with the symplectic group of index matrices on the canonical skew matrix

$$\mathbf{G} = \begin{bmatrix} . & 1 & . & . \\ 2 & . & . & . \\ . & . & . & 1 \\ . & . & 2 & . \end{bmatrix}$$

whose elements belong to $GF(3)$. Zeros are indicated by dots.

Generators of CS may, by (**3**), be taken as

$$\mathbf{Q} = \begin{bmatrix} 1 & 2 & 1 & . \\ 1 & . & 1 & . \\ . & 1 & . & 2 \\ . & . & 1 & . \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 1 & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$$

of periods 10 and 3 respectively, with defining relation $(\mathbf{QD})^5 = \mathbf{I}$. It is easy to verify that $\mathbf{Q}$ and $\mathbf{D}$ transform the basic Clifford set into the respective Clifford sets

$$W^{10}_{10},\ W^{01}_{10},\ W^{01}_{11},\ W^{00}_{11},\ W^{20}_{00}$$

and

$$W^{10}_{00},\ W^{20}_{10},\ W^{11}_{10},\ W^{11}_{11},\ W^{10}_{11}.$$

In passing, we remark that CS is identical with the *groupe abélien* of Jordan (**4**) (the *Abelian linear group* of Dickson (**5**)).

Denote by CT the *Clifford similarity* (*transform*) *group*, namely, the group of similarity transform matrices R (non-singular) which carry the basic Clifford set into another set of CG. Elements of R may be $0, 1, \epsilon, \epsilon^2$. Just one element of CS corresponds to each element of CT, whereas to

each element of CS correspond 81 elements of CT: that is, there is a 1–81 correspondence between CS and CT. To the unit element of CS corresponds the self-conjugate subgroup CG of CT, and CS is the factor-group of CT. By (3), the order of CT is $51840 \times 81 = 4\,199\,040$.

Operations **D** and **Q** which generate CT are of periods 3 and 10 respectively. By (3) the former may be taken as the diagonal matrix $(1\,1\,\epsilon^2\,1\,1\,\epsilon^2\,1\,1\,\epsilon^2)$, which is expressible as $I+W^{10}_{00}+\epsilon W^{20}_{00}$, apart from a scalar factor. The form of the other generator is deferred for the moment.

Included in the group of index matrices CS there is the special matrix $2\mathbf{I} = \mathbf{Q}^5$ to which corresponds in CT a set of 81 transform matrices. Let J be one of these. The most satisfactory form of J is found (3) to be that given by the substitution matrix of period 2 whose only non-zero elements are $j^{a\;\,a'}_{2a\,2a'} = 1$. Clearly, J corresponds to the substitution 132 798 465. It is symmetric, involutory, and orthogonal. Further,

$$\begin{aligned} JW^{aa'}_{bb'}J &= W^{2a\,2a'}_{2b\,2b'} \\ &= \epsilon^{2(ab+a'b')}(W^{aa'}_{bb'})^2, \quad \text{from (ii).} \end{aligned} \tag{v}$$

That is, J transforms each W into its square, apart from a scalar factor.

Non-zero elements of $JW^{aa'}_{bb'}$ can be compactly represented as

$$\alpha^{2c\;\;\;2c'}_{c+b\,c'+b'} = \epsilon^{ac+a'c'} \quad (c,\,c' = 0,\,1,\,2).$$

The 81 matrices JW do not form a group since, by (v), the group property of closure is obviously lacking. However, by virtue of the relation between CS and CT, the W and JW form a self-conjugate subgroup of CT.

For Q, the form required is such that $Q^5 = J$, so that $Q^{10} = J^2 = I$. This form is not unique since we may also use QW. Experimentation shows that the most convenient form of Q (due to T. G. Room) is that whose general element is given by

$$\alpha^{aa'}_{bb'} = -\tfrac{1}{3}\epsilon^{a^2+ab+a'b'+2a'b}.$$

2. Invariant spaces of the involutions JW

Each collineation matrix JW is a harmonic inversion with respect to a solid and a [4]. If the solid $\Sigma^{aa'}_{bb'}$ and the [4], $\Pi^{aa'}_{bb'}$, are associated with the matrix $JW^{aa'}_{bb'}$, the 81 pairs of invariant spaces are determined thus:

$$\Pi^{aa'}_{bb'} \text{ by the 4 primes } x_{cc'} = \epsilon^{2[a(b+c)+a'(b'+c')]}x_{b+2c\,b'+2c'},$$

$$\Sigma^{aa'}_{bb'} \text{ by the 4 points } A_{cc'}-\epsilon^{a(b+c)+a'(b'+c')}A_{b+2c\,b'+2c'} \quad (c,\,c' = 0,\,1,\,2).$$

It is found that 9 [4]'s pass through each vertex of the simplex of reference, and 9 solids lie in each prime face of the simplex.

Two [4]'s, $\Pi^{aa'}_{bb'}$ and $\Pi^{\alpha\alpha'}_{\beta\beta'}$, determined respectively by

$$\left.\begin{aligned} x_{cc'} &= \epsilon^{2u} x_{b+2c\, b'+2c'} \\ x_{\gamma\gamma'} &= \epsilon^{2v} x_{\beta+2\gamma\, \beta'+2\gamma'} \end{aligned}\right\}, \tag{vi}$$

and

where

$$u = a(b+c)+a'(b'+c'), \qquad v = \alpha(\beta+\gamma)+\alpha'(\beta'+\gamma'),$$

intersect in a space satisfying

$$x_{cc'} = \epsilon^{2w} x_{\beta+2b+c, \beta'+2b'+c'}, \tag{vii}$$

where $$w = u+\alpha(\beta+b+2c)+\alpha'(\beta'+b'+2c').$$

In equations (vii), there are always exactly 6 independent equations, namely, 3 pairs of which one pair links x_{00} with $x_{\beta+2b,\beta'+2b'}$ with $x_{2\beta+b,2\beta'+b'}$, and the others are of the same type. Taken with either set of equations (vi), equations (vii) determine the *line* of intersection of the two [4]'s.

3. A configuration in [8]

Spaces Π give rise to a configuration in [8] which is invariant under the group CT. Included in this configuration are 12 basic planes,

$$\begin{array}{lll} A_{00}A_{10}A_{20}, & A_{01}A_{11}A_{21}, & A_{02}A_{12}A_{22}, \\ A_{00}A_{01}A_{02}, & A_{10}A_{11}A_{12}, & A_{20}A_{21}A_{22}, \\ A_{00}A_{11}A_{22}, & A_{01}A_{12}A_{20}, & A_{02}A_{10}A_{21}, \\ A_{00}A_{12}A_{21}, & A_{01}A_{10}A_{22}, & A_{02}A_{11}A_{20}. \end{array}$$

(The suffixes of the A's form the pattern given by Clebsch (**6**) in his representation of inflexion triangles in the plane.)

Our configuration is built up from the simplest [4], Π^{00}_{00}, given by $x_{ij} = x_{2i,2j}$ $(i, j = 0, 1, 2)$. This passes through A_{00} and is determined by A_{00} and the 4 points

$$B_1 = A_{10}+A_{20}, \quad B_2 = A_{01}+A_{02}, \quad B_3 = A_{11}+A_{22}, \quad B_4 = A_{12}+A_{21}.$$

Altogether, Π^{00}_{00} contains 40 points, namely

$$A \equiv A_{00}, \tag{1}$$

$$A+\epsilon^{\rho}B_i \quad (\rho = 0, 1, 2;\ i = 1, 2, 3, 4), \tag{12}$$

$$A+\epsilon^{\alpha}B_1+\epsilon^{\beta}B_2+\epsilon^{\gamma}B_3+\epsilon^{\delta}B_4, \tag{27}$$

where $\alpha+\beta+\gamma+\delta \equiv 0 \pmod 3$ $(\rho, \alpha, \beta, \gamma, \delta = 0, 1, 2)$. These points are collinear in sets of 4 on 40 lines with 4 lines through each point. Through each point there also pass 9 Π-spaces: for example, the 9 spaces $\Pi^{a}_{2a+b+1},{}^{b}_{a+2}$ have in common the point

$$(1\,\epsilon\,\epsilon^2, \epsilon\,\epsilon\,\epsilon, \epsilon\,1\,\epsilon^2).$$

Now, there are 40 points in each space, 81 such spaces, and 9 of these spaces through each point. Thus, in the configuration there are $\frac{1}{9}(40\times 81) = 360$ points, namely,

vertices of the simplex of reference, (9)

9 other critic centres in each of the 12 basic planes, (108)

others, (243).

The *critic centres* of a triangle ABC are, by (7), the 12 points

$$A,\ B,\ C,\ A+\epsilon^{\rho}B+\epsilon^{\sigma}C \quad (\rho, \sigma = 0, 1, 2).$$

They provide a way of forming the *Jacobian configuration* in the plane: that is, the configuration based on the 9 inflexions of a plane canonical cubic curve. Following Baker (8), I shall call a plane containing the Jacobian configuration a *Jacobian plane.*

The 360 points lie by sets of 12, the critic centres, in a set of planes of which the 12 basic planes are typical. Four planes pass through a point: for example, in a simple instance, the 4 planes through A_{00}. In all, there are $\frac{1}{12}(360\times 4) = 120$ planes.

Through each line pass three Π-spaces. For example, the 3 spaces Π_{10}^{00}, Π_{11}^{21}, Π_{12}^{12} have in common the line joining the points

$$(1\,1\,.,\epsilon^2\,1\,.,1\,\epsilon^2\,.) \quad \text{and} \quad (.\,.\,1,\,.\,.\,\epsilon,\,.\,.\,\epsilon).$$

Bearing in mind the fact that each of the 81 Π's contains 40 lines, we deduce that the totality of lines in the configuration is $\frac{1}{3}(40\times 81) = 1080$.

The 243 points mentioned are found to be of the form

$$(1\,\epsilon^{\rho}\epsilon^{\sigma},\epsilon^{\alpha}\epsilon^{\beta}\epsilon^{\gamma},\epsilon^{\lambda}\epsilon^{\mu}\epsilon^{\nu}) \quad (\rho, \sigma, \alpha, \beta, \gamma, \lambda, \mu, \nu = 0, 1, 2),$$

where
$$\rho+\sigma \equiv \alpha+\beta+\gamma \equiv \lambda+\mu+\nu \pmod 3,$$
$$\alpha+\lambda \equiv \rho+\beta+\mu \equiv \sigma+\gamma+\nu \pmod 3.$$

Here, there are 3 independent equations, say,

$$\rho \equiv \lambda+\mu+\nu-\sigma \pmod 3,$$
$$\beta \equiv \mu-\sigma+\gamma-\lambda \pmod 3,$$
$$\alpha \equiv \gamma-\lambda+\sigma+\nu \pmod 3,$$

showing that the variations of the parameters produce $3^{8-3} = 3^5 = 243$ points.

From the point of view of group theory, the configuration is seen to be quite general and not dependent on the choice of Π_{00}^{00} as [4] from which it is constructed. Firstly, we prove the theorem:

THEOREM 1. *A matrix of CT leaves Π_{00}^{00} invariant only if it commutes with J.*

Let R be any matrix of CT. In the set of 81 matrices $W_{bb'}^{aa'}R$ there is only one which commutes with J (proved in Theorem 7). Suppose that this is

$$R = (r_{tt'}^{ss'}).$$

Then, since

$$JRJ = (r_{2t\,2t'}^{2s\,2s'}),$$

it follows that

$$r_{tt'}^{ss'} = r_{2t\,2t'}^{2s\,2s'}.$$

Also, Π_{00}^{00}, determined by $x_{ij} = x_{2i\,2j}$, is clearly invariant under R.

Next, take $W_{bb'}^{aa'}R$ and a general point x_{ij}. From the definition of $W_{bb'}^{aa'}$, we have

$$W_{bb'}^{aa'}x_{ij} = \epsilon^{ai+a'j}x_{b+i\,b'+j}.$$

Therefore

$$W_{bb'}^{aa'}Rx_{ij} \neq x_{2i\,2j}$$

unless a, a', b, b' are all zero: that is, W is the unit matrix.

Hence we have the result.

Later (Theorem 7) it is proved that matrices commuting with J are the matrices of CS as a subgroup of CT.

Remembering that to **R** of CS there correspond the RW of CT, we are thus led to consider the configuration of spaces $W\Pi_{00}^{00}$ which is invariant under CT. Any property which the configuration has in relation to a point P, it has also in relation to the point XP, where X is any matrix of CT.

It is therefore permissible for us to take A_{00} as a starting point. The configuration which has been built up on the basis of A_{00} and Π_{00}^{00} thus consists of

all points XA_{00},

all lines $X(A_{00}, A_{10}+A_{20})$,

all planes $X(A_{00}, A_{10}, A_{20})$,

all [4]'s $X(A_{00}, A_{10}+A_{20}, A_{01}+A_{02}, A_{11}+A_{22}, A_{12}+A_{21})$.

Unifying these remarks, we have the theorem:

THEOREM 2. *In* [8] *there is a configuration, invariant under the operations of CT, consisting of* 360 *points,* 1080 *lines,* 120 *Jacobian planes, and* 81 [4]*'s such that there are* 4 *points on each line,* 12 *points in each plane,* 40 *points in each* [4]; 12 *lines through each point,* 40 *lines in each* [4]; 4 *planes through a point*; 9 [4]*'s through each point,* 3 [4]*'s through a line.*

4. Determinantal cubic primals

Consider the cubic primal V_7^3 defined determinantally as

$$V_0 \equiv \begin{vmatrix} x_{00} & x_{10} & x_{20} \\ x_{01} & x_{11} & x_{21} \\ x_{02} & x_{12} & x_{22} \end{vmatrix} = 0.$$

From this are derived 5 similar loci V_i ($i = 1, 2, 3, 4, 5$) by means of the collineations ($i, j = 0, 1, 2$)

$$\begin{array}{lll} x'_{ij} = x_{i+j,j}, & \text{i.e.} \quad x' = Kx & \text{for } V_1, \\ x'_{ij} = x_{i+2j,j}, & \text{i.e.} \quad x' = K^2x & \text{for } V_2, \\ x'_{ij} = x_{i,2i+j}, & \text{i.e.} \quad x' = Lx & \text{for } V_3, \\ x'_{ij} = x_{i,i+j}, & \text{i.e.} \quad x' = L^2x & \text{for } V_4, \\ x'_{ij} = x_{i+j,i+2j}, & \text{i.e.} \quad x' = KL^2x & \text{for } V_5, \end{array}$$

where K, L correspond to the substitutions 123 564 978 and 186 429 753 respectively, with $K^3 = L^3 = (KL)^6 = I$. These V_i are not all linearly independent since, for example, $V_0+V_3+V_4 = 0$.

It may be verified that the V_i are left invariant by the matrices JW and, further, that Σ_{00}^{00} (and all the solids Σ) lie on all of them.

Thus we have the theorem:

THEOREM 3. *In* [8] *there exists a set of* 6 *determinantal cubic primals, not all linearly independent, having in common the* 81 *solids invariant under the collineation matrices* JW.

Geometrical properties of determinantal loci of the type V_i are well known. Our V_0 is, for example, the *key-manifold* of the $|3, 3|$ series dealt with by Room (9). Lying on it are two families of ∞^2 generating [5]'s and one family of ∞^4 generating [4]'s. The generating [4]'s are tangent spaces to a sextic fourfold locus M_4^6 occurring as a double locus on V_0, and M_4^6 is thus invariant under the collineations JW. Further, M_4^6 has two ∞^2 families of generating planes. The transforms of a point on M_4^6 are the 81 intersections of 9 generating planes of one system with 9 generating planes of the other system. Each plane is a Jacobian plane.

5. An invariant locus in [8]

Next it is shown that there is a locus in [8] which is invariant under the collineations of CT. Consider the 5 linearly independent cubic primals

$$\theta\colon \sum_{i,j=0}^{2} x_{ij}^3 = 0,$$

$$\phi_1\colon x_{00}x_{01}x_{02}+x_{10}x_{11}x_{12}+x_{20}x_{21}x_{22} = 0,$$

$$\phi_2\colon x_{00}x_{10}x_{20}+x_{01}x_{11}x_{21}+x_{02}x_{12}x_{22} = 0,$$

$$\phi_3\colon x_{00}x_{11}x_{22}+x_{01}x_{12}x_{20}+x_{02}x_{10}x_{21} = 0,$$

$$\phi_4\colon x_{00}x_{12}x_{21}+x_{01}x_{10}x_{22}+x_{02}x_{11}x_{20} = 0.$$

The forms of ϕ_i ($i = 1, 2, 3, 4$) occur as the 4 possible arrangements of the 9 coordinates in 3 sets of 3, the pattern used by Clebsch (**6**) in another connexion. It is noted that $\phi_1+\phi_2+\phi_3+\phi_4 = 0$ is called *Hesse's equation* by Jordan (**10**).

Any point P, with coordinates a_{ij}, and its 81 transforms JWP, all lie on the cubic primal

$$\theta - \sum_{i,j=0}^{2} a_{ij}^3 \frac{(\phi_1+\phi_2+\phi_3+\phi_4)}{K} = 0,$$

where K is the term obtained from $\phi_1+\phi_2+\phi_3+\phi_4$ on replacing x_{ij} by a_{ij}. That is, each cubic primal of the pencil

$$\theta-\lambda(\phi_1+\phi_2+\phi_3+\phi_4) = 0 \quad (\lambda \text{ a parameter})$$

is self-transformed by the operations of the 81 JW.

The 5 primals θ and ϕ_i ($i = 1, 2, 3, 4$) have a common locus, say L, which is shown to be invariant under all the collineations of the group CT.

In (**3**), it is proved that CT may be generated by the matrices $\Omega_{ss'}^{rr'}$, one being associated with each W. These generators are defined as

$$\Omega = \frac{1}{\sqrt{(-3)}}(I+W+\epsilon W^2) \quad \text{with } \Omega^3 = I.$$

For instance, the generator D is expressible as $-\epsilon^2\Omega_{00}^{10}$.

If the locus is invariant under all the Ω, then it is invariant under all the operations of CT. Since

$$(W_{bb'}^{aa'})^2 = \epsilon^{ab+a'b'} W_{2b\,2b'}^{2a\,2a'},$$

we have

$$\Omega_{bb'}^{aa'} = \frac{1}{\sqrt{(-3)}}(I+W_{bb'}^{aa'}+\epsilon^g W_{2b\,2b'}^{2a\,2a'}),$$

where $g = 1+ab+a'b'$. Picking out the elements belonging to the row numbered rr', we find that $\Omega_{bb'}^{aa'}$ transforms the prime $x_{rr'} = 0$ into the prime

$$x_{rr'}+\epsilon^{ar+a'r'}x_{r+b\,r'+b'}+\epsilon^{g+2ar+2a'r'}x_{r+2b\,r'+2b'} = 0.$$

Using this equation in conjunction with the identity

$$\epsilon(x^3+y^3+z^3-3\epsilon xyz) \equiv (\epsilon x+y+z)(x+\epsilon y+z)(x+y+\epsilon z),$$

we find that the 4 cubic forms ϕ_i and the 12 cubic forms

$$\psi_{pi} \equiv \theta - 3\epsilon^p \phi_i = 0 \quad (i = 1, 2, 3, 4;\ p = 1, 2, 3)$$

are transformed into one another by all the operations of CT. Thus we have the theorem:

THEOREM 4. *The locus L common to the cubic primals θ and ϕ_i is invariant under all the operations of the group CT.*

The 6 determinantal loci V_i ($i = 0, 1, 2, 3, 4, 5$) are now seen to be differences of the 4 ϕ_i ($i = 1, 2, 3, 4$), namely

$$V_0 = \phi_3 - \phi_4,\ V_1 = \phi_4 - \phi_1,\ V_2 = \phi_1 - \phi_3,\ V_3 = \phi_2 - \phi_3,\ V_4 = \phi_4 - \phi_2,$$
$$V_5 = \phi_1 - \phi_2.$$

Only 3 of the V are linearly independent.

Consequently, the locus common to the ϕ alone is the common locus of the V: that is, the 81 solids Σ are invariant under the transformations of CT and not merely, as in Theorem 3, under the collineations of the JW. But these solids also lie on θ, as may be verified, so that they form part of the locus L.

Extending Theorem 3, we thus have the theorem:

THEOREM 5. *The* 81 *solids Σ belong to the locus L.*

The $108+243 = 351$ points occurring in the configuration of § 3 (vertices of the simplex of reference being excluded) do not lie on L but are incident on the ψ in a certain manner.

Finally, it may be noted that, corresponding to each

$$\psi_{pi} \equiv \theta - 3\epsilon^p \phi_i = 0,$$

there is a

$$\theta_{pi} \equiv 3(\theta + 6\epsilon^p \phi_i) = 0.$$

6. Intersection of the solids on L

Now Σ_{00}^{00} is determined by the 4 points

$$P \equiv A_{10} - A_{20},\quad Q \equiv A_{01} - A_{02},\quad R \equiv A_{11} - A_{22},\quad S \equiv A_{21} - A_{12},$$

and lies in the prime $x_{00} = 0$. In Σ_{00}^{00} lie 40 points, namely,

$$P,\ Q,\ R,\ S,\ Q+\epsilon^\rho R+\epsilon^\sigma S,\ P+\epsilon^\rho R-\epsilon^\sigma S,$$
$$P-\epsilon^\rho Q+\epsilon^\sigma S,\ P+\epsilon^\rho Q-\epsilon^\sigma R \qquad (\rho, \sigma = 0, 1, 2).$$

Through each point there pass just 3 solids, as we can verify. For example, $Q+\epsilon R+S$ lies in the 3 solids $\Sigma_{00}^{00}, \Sigma_{10}^{21}, \Sigma_{20}^{12}$.

Each of the 4 sets of 12 points

$$Q,\ R,\ S,\ Q+\epsilon^{\rho}R+\epsilon^{\sigma}S,$$
$$P,\ R,\ S,\ P+\epsilon^{\rho}R-\epsilon^{\sigma}S,$$
$$P,\ Q,\ S,\ P-\epsilon^{\rho}Q+\epsilon^{\sigma}S,$$
$$P,\ Q,\ R,\ P+\epsilon^{\rho}Q-\epsilon^{\sigma}R,$$

is coplanar, forming a Jacobian configuration. Thus, in Σ_{00}^{00}, the intersections with other solids form 4 Jacobian planes constituting a tetrahedron whose vertices are P, Q, R, S: that is, the 4 points determining the space Σ_{00}^{00}.

Each of the 81 solids Σ contains 40 points, 3 solids passing through each point. Arguing as for the Π-spaces in § 3, and synthesizing the configuration from Σ_{00}^{00}, we conclude that the number of points of intersection of the solids is $\frac{1}{3}(40\times 81) = 1080$. Of these, 108 are vertices of the 81 tetrahedra since each tetrahedron has 4 vertices and each vertex lies in 3 solids. Coordinates of these vertices are

$$(1-\epsilon^{a}\,.,\ .\ .\ .\ .,\ .\ .\ .) \quad (a = 0,\ 1,\ 2),$$

if we count all the possible permutations. Passing to the remaining $1080-108 = 972$ points, we find that they can be grouped into 12 sets of 81. Such a set is $(.\ .\ .,1\,\epsilon^{\alpha}\,\epsilon^{\beta}, -\epsilon^{\lambda}-\epsilon^{\mu}-\epsilon^{\nu})$ with

$$\alpha+\beta \equiv \lambda+\mu+\nu \pmod 3).$$

Zero coordinates in each of the 12 sets positionally obey the Clebsch pattern introduced in § 3.

Grouping together these results, we have the theorem:

THEOREM 6. *On L, the* 81 *solids (each containing* 4 *Jacobian planes) intersect in* 1080 *points, with* 40 *points in each solid and* 3 *solids through each point.*

Using Perazzo primals (introduced in § 9), we can deduce the existence of 324 other Jacobian planes lying on L, but I do not prove this statement here.

7. The Burkhardt configuration in [4] and the symplectic group of order 25920

I prove the theorem:

THEOREM 7. *As a subgroup of CT, CS (of order* 51840*) consists of those elements of CT which commute with J.*

Firstly, I demonstrate that in any set of 81 matrices $XW_{bb'}^{aa'}$ there is one and only one which commutes with J. For, suppose X one of the

matrices corresponding to **X**. It thus corresponds also to (2**I**)**X**(2**I**). Therefore, there exist a $W_{bb'}^{aa'}$ and a ρ (= 0, 1, 2) for which

$$\epsilon^\rho X W_{bb'}^{aa'} = JXJ:$$

that is,

$$\begin{aligned}\epsilon^\rho X(W_{bb'}^{aa'})^2 &= JXJW_{bb'}^{aa'}\\ &= \epsilon^{2(ab+a'b')}JX(W_{bb'}^{aa'})^2J, \quad \text{by § 1 (v)},\end{aligned}$$

whence

$$X(W_{bb'}^{aa'})^2J = JX(W_{bb'}^{aa'})^2$$

apart from a possible exponential factor. Consequently $X(W_{bb'}^{aa'})^2$ commutes with J. Since the set of 81 matrices was arbitrary, the result follows.

Now

$$\begin{aligned}J(JD^2JD^2)J &= J(JD^2)^2J\\ &= J(D^2J)^2J, \quad \text{since } (JD^2)^2 = (D^2J)^2 = C, \text{ say},\\ &= JD^2JD^2JJ\\ &= JD^2JD^2.\end{aligned}$$

Therefore, JD^2JD^2 which corresponds to (2**I**)**D**^{2}(2**I**)**D**^{2} (that is, to **D**) commutes with J. It is easy to show that C, which may also be written as $(DJD)^2$, consists entirely of diagonal elements $(1\,\epsilon\,\epsilon, 1\,\epsilon\,\epsilon, 1\,\epsilon\,\epsilon)$ and has period 3. Further, Q commutes with J since $Q^5 = J$. Hence we have the theorem:

THEOREM 8. *As a group of matrices in* [8], *CS can be generated by Q and the diagonal matrix C.*

Eighty other groups like CS exist, namely, the groups of matrices of CT which commute with the JW (other than J). Each solid Σ and its associated [4], Π, will remain invariant under the appropriate subgroup CS of CT of order 51840, while the remaining solids and [4]'s will be permuted amongst themselves.

Next, we prove the theorem:

THEOREM 9. *If A and B* (*elements of CT*) *induce identical collineations in* Π_{00}^{00}, *then* $B = JA$.

By hypothesis, the linear transformations A, B (of CT) induce identical collineations in Π_{00}^{00}. Therefore, AB^{-1} leaves Π_{00}^{00} point-by-point invariant. Consequently,

$$AB^{-1} = J,$$

i.e.

$$JB = A,$$

which is

$$JA = B,$$

so that the theorem is established.

This means that A, JA represent the same collineation in the space. In terms of the index matrix elements of the symplectic group, this shows that we must identify **A** and $2\mathbf{IA} = 2\mathbf{A}$. From Theorems 1, 7, and 9, we deduce that

(*a*) the matrices of CS alone preserve the invariance of Π_{00}^{00};

(*b*) the group of collineations in Π_{00}^{00} is $\frac{1}{2}CS$ (of order 25920), which corresponds to the group obtained from the symplectic group by identifying the matrices **X**, 2**X**.

Now consider the section of the locus L by Π_{00}^{00}. This consists of the points common to the primals in [4]

$$\theta'\colon\ x^3+2(y_1^3+y_2^3+y_3^3+y_4^3) = 0,$$
$$\phi_1'\colon\ xy_2^2+2y_1y_3y_4 = 0,$$
$$\phi_2'\colon\ xy_1^2+2y_2y_3y_4 = 0,$$
$$\phi_3'\colon\ xy_3^2+2y_1y_2y_4 = 0,$$
$$\phi_4'\colon\ xy_4^2+2y_1y_2y_3 = 0.$$

(As coordinates in Π_{00}^{00} I have chosen $x = x_{00}$, $y_1 = x_{10} = x_{20}$, $y_2 = x_{01} = x_{02}$, $y_3 = x_{11} = x_{22}$, $y_4 = x_{12} = x_{21}$.)

These primals have in common only 45 points, namely

$$(.\,;1-\epsilon^a\,.\,.)\quad (a = 0, 1, 2) \qquad (18)$$

and $$(-2;\epsilon^t\epsilon^u\epsilon^v\epsilon^w)\quad \text{with } t+u+v+w \equiv 0 \pmod 3 \qquad (27),$$

wherein the semicolon separates the first (fixed) coordinate from the remaining 4, and both 1 and $-\epsilon^a$ range over the last 4 positions. All these points are simple. That the [4], Π_{00}^{00}, cuts L in 45 points suggests that L is of order 45 and dimension 4, a surmise which is confirmed in § 9.

Every matrix of CT which commutes with J, since it leaves L and Π_{00}^{00} invariant, provides a collineation of Π_{00}^{00} permuting these 45 points. That is, by result (*b*) above, $\frac{1}{2}CS$ is a permutation group on these 45 points.

Clearly, the set of 18 points lies in the prime $x = 0$ and forms the Jacobian configuration (of inflexions) in each of the 4 planes $x = 0$, $y_i = 0$ $(i = 1, 2, 3, 4)$. Under the operations of $\frac{1}{2}CS$, this solid is transformed into other solids containing corresponding sets of 18 points.

In number, these solids are 40, namely,

$$x = 0 \qquad (1),$$
$$x+2\epsilon^a y_i = 0 \quad (a = 0, 1, 2;\ i = 1, 2, 3, 4) \qquad (12),$$
$$x+2(\epsilon^t y_1+\epsilon^u y_2+\epsilon^v y_3+\epsilon^w y_4) = 0 \qquad (27),$$

where $t+u+v+w \equiv 0 \pmod 3$. These equations may be compared with the forms for the 40 points in Π^{00}_{00} (§ 3).

Generators of this group in Π^{00}_{00}, under which the 45 points and 40 solids are invariant, are the reduced forms of C and Q obtained by adding together the columns corresponding to the coordinates which have been equated and suppressing one of each pair of corresponding rows. Thus

$$Q^* = -\tfrac{1}{3}\begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ \epsilon & -\epsilon & 2\epsilon & -\epsilon & -\epsilon \\ 1 & -1 & -1 & 2 & -1 \\ \epsilon & 2\epsilon & -\epsilon & -\epsilon & -\epsilon \\ \epsilon & -\epsilon & -\epsilon & -\epsilon & 2\epsilon \end{bmatrix} \quad \text{with } Q^{*5} = I$$

and C^* (with $C^{*3} = I$) is the diagonal matrix $(1\,\epsilon, 1\,\epsilon\,\epsilon)$. Of course, the original J reduces to the unit matrix.

The configuration of the 45 points may conveniently be called the *Clifford-derived configuration*. This is now compared with the *Burkhardt configuration*, both configurations being in [4]. The *Burkhardt primal* [Burkhardt (**11**)] of order 4, B^4_3, and its associated configuration formed from the 45 nodes lying on it have been extensively studied by Baker (**8**). It is known [Baker (**8**), Todd (**12**)] that the Burkhardt primal is a rational locus.

Six homogeneous coordinates x_i ($i = 1,\dots, 6$) with the proviso that $\sum x = 0$ are used by Baker in his description of the geometry of the primal. Notwithstanding the great value of this approach, it is more convenient for us to suppress a redundant coordinate and label the remaining ones in order z_i ($i = 0, 1, 2, 3, 4$). To facilitate reference, I shall associate Burkhardt's name with this notation. Suppressing x_2 (say), we find that the 45 Burkhardt nodes are

$$(1\,\epsilon, \epsilon\,\epsilon^2\,\epsilon^2) \quad (30),$$

$$(1\,., \ldots) \quad (5),$$

$$(1\,-1, \ldots) \quad (10),$$

counting all the possible permutations of the coordinates.

Solids (called *Steiner solids* by Baker) become

$$z_i+z_j+z_k = 0 \qquad (10),$$

$$z_i = \epsilon^a z_j \qquad (20),$$

$$-\epsilon^a z_0+z_1+z_2+z_3+z_4 = 0 \quad (10).$$

To identify the Burkhardt and Clifford-derived configurations, a transformation is required which will connect the two sets of coordinates z_i and x, y_1, y_2, y_3, y_4. One such possibility is

$$\begin{bmatrix} x \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -2\epsilon & 2 & 2 & 2 & 2 \\ \epsilon & -1 & -\epsilon^2 & -\epsilon^2 & -\epsilon^2 \\ \epsilon & -\epsilon^2 & -1 & -\epsilon^2 & -\epsilon^2 \\ \epsilon & -\epsilon^2 & -\epsilon^2 & -1 & -\epsilon^2 \\ \epsilon & -\epsilon^2 & -\epsilon^2 & -\epsilon^2 & -1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix},$$

which converts the vertices of the simplex of reference in the Burkhardt set into 5 points of the Clifford-derived configuration.

The formal identification of the two configurations is then established. Thus in the Clifford-derived configuration there are the elements corresponding to the following Burkhardt configuration elements:

45 nodes,
240 lines,
40 Jacobian planes,
40 Steiner solids,

and their various interrelationships as given by Baker (**8**). Thus we have the theorem:

THEOREM 10. *The section of L by a* [4] *is a set of* 45 *points forming the Burkhardt configuration.*

Now, the group of operations permuting amongst themselves the lines of a cubic surface in [3] is known to be of order 51840 [Jordan (**10**)]. The subgroup of this, of order 25920 and index 2, was proved simple by Jordan (loc. cit.), and was considered by him as a group of permutations of tritangent planes of the cubic surface containing only the even permutations. This is the group which Baker (**8**) uses in his study of the Burkhardt primal and Burkhardt configuration, and which leaves the primal invariant. This simple group of order 25920 has been closely investigated by Todd (**12**) and (**13**), in which several pairs of generators appear. (Compare these with our Q^* and C^*.) Coble (**14**) and (**15**), among others, has contributed considerably to the theory of this group.

Identification of the Burkhardt and Clifford-derived configurations under the operations of $\frac{1}{2}CS$ leads us to infer that

$\frac{1}{2}CS$ *is the cubic surface (sub)group.*

Such an isomorphism is mentioned by Dickson (**5**) with a different notation.

Besides being generated by the two reduced matrix operators Q^* and

C^*, $\frac{1}{2}CS$, being the group of collineations in Π_{00}^{00} corresponding to the subgroup of the symplectic group CS for which $\mathbf{X}$ and $2\mathbf{X}$ are identified, can also be generated by $\mathbf{D}$ and $2\mathbf{Q}$ since $\mathbf{D}^3 = \mathbf{I}$ and $\mathbf{Q}^5 = 2\mathbf{I}$.

Since CS is the symplectic group of matrices of size 4 having elements in $GF(3)$, the reduced group $\frac{1}{2}CS$ is the group of projective transformations of [3] which leaves invariant a linear complex. Expressed otherwise we can say:

The simple group of order 25920 *associated with the lines of a cubic surface in* [3] *is isomorphic with the group of collineations in* [3], *having coordinates in* $GF(3)$, *which leaves a linear complex invariant.*

Edge (**16**) has obtained results, for geometry in [3] over $GF(3)$, which may be regarded as complementary to the above results.

8. Involutions and the Burkhardt configuration

With each node of the Burkhardt configuration there is associated a definite solid called the *polar prime* or, following Baker (**8**), the *Jordan prime* of the node. Converting Baker's notation to ours, we find that the 45 Jordan primes have the prime coordinates hereunder:

$$(1 - \epsilon^t - \epsilon^u - \epsilon^v - \epsilon^w) \quad \text{with } t+u+v+w \equiv 0 \pmod 3 \quad (27),$$

$$(.\,; 1 - \epsilon^a\ .\ .) \quad (a = 0, 1, 2) \qquad (18).$$

Baker (loc. cit.) has shown that the configuration is left invariant by the harmonic inversion with respect to any node, say A, and its Jordan prime. Such a harmonic inversion will be denoted by $p(A)$ and, again following Baker, will be called the *projection* from the node A. It is the operation of the group $\frac{1}{2}CS$ which leaves invariant A and the 12 nodes in the Jordan prime of A, and which interchanges in pairs the two nodes other than A, on each of the 16 *κ-lines* through A. Consequently, $p(A)$ performs an even permutation on the 45 nodes. The projections generate the group $\frac{1}{2}CS$, and Burkhardt's own 4 generators are expressible in terms of them. Further, Todd (**13**) shows that the 45 projections are conjugate, and form a complete conjugate set in $\frac{1}{2}CS$, and, moreover that every operation of $\frac{1}{2}CS$ is expressible as a product of no more than 5 projections. He classifies these operations, obtains their periods, and determines the conjugate sets within $\frac{1}{2}CS$. Todd also examines various pairs of generators of $\frac{1}{2}CS$ having fairly small periods.

Jordan primes in Π_{00}^{00} are now shown to lie on L. To do this, we observe that each of the Jordan primes contains at least two linearly independent points of the type $(.\,; 1-\epsilon^a\ .\ .)$ and two linearly independent

points of the type $(-2; \epsilon^t \epsilon^u \epsilon^v \epsilon^w)$, where $t+u+v+w \equiv 0 \pmod 3$. Originally, in [8], before reduction, these were the points

$$(.\,1\,1, -\epsilon^a\,.\,., -\epsilon^a\,.\,.)$$

and
$$(-2\,\epsilon^t \epsilon^t, \epsilon^u \epsilon^v \epsilon^w, \epsilon^u \epsilon^w \epsilon^v),$$

with the usual restriction on t, u, v, w. Substitute such coordinates in the equations determining L and it is found that each of the 45 Jordan primes lies on L. It is simple to verify that they are solids distinct from the 81 solids Σ. Furthermore, it can be shown that the Jordan solids have no points in common with the Σ. Hence we have the theorem:

THEOREM 11. *The locus L contains* 45 *Jordan solids not intersecting any of the* 81 Σ.

Throughout the investigations of Baker and Todd mentioned above, the matrix forms of the projections are implicit. Explicit forms for these projections are now given in terms of Clifford matrices.

Suppose, for convenience of notation, that only indices of ϵ are written down so that 0, 1, 2 represent 1, ϵ, ϵ^2, respectively. In addition, represent $2\epsilon^{-a}$ by $2(-a)$. Then the reduced forms of the 27 involutory matrices in [4] yielding the projections $p(-2; \epsilon^t \epsilon^u \epsilon^v \epsilon^w)$, where

$$t+u+v+w \equiv 0 \pmod 3,$$

are

$$P^* = \begin{bmatrix} 0 & 2(-t) & 2(-u) & 2(-v) & 2(-w) \\ t & 2(0) & -(t-u) & -(t-v) & -(t-w) \\ u & -(u-t) & 2(0) & -(u-v) & -(u-w) \\ v & -(v-t) & -(v-u) & 0 & -(v-w) \\ w & -(w-t) & -(w-u) & -(w-v) & 2(0) \end{bmatrix}$$

apart from an appropriate scalar factor chosen so that $(P^*)^2 = I$.

Correspondingly, in [8] the matrix is

$P =$

$$\begin{bmatrix} 0 & -t & -t & -u & -v & -w & -u & -w & -v \\ t & 0 & 0 & t-u+1 & t-v+1 & t-w+1 & t-u+2 & t-w+2 & t-v+2 \\ t & 0 & 0 & t-u+2 & t-v+2 & t-w+2 & t-u+1 & t-w+1 & t-v+1 \\ u & u-t+2 & u-t+1 & 0 & u-v+2 & u-w+1 & 0 & u-w+2 & u-v+1 \\ v & v-t+2 & v-t+1 & v-u+1 & 0 & v-w+2 & v-u+2 & v-w+1 & 0 \\ w & w-t+2 & w-t+1 & w-u+2 & w-v+1 & 0 & w-u+1 & 0 & w-v+2 \\ u & u-t+1 & u-t+2 & 0 & u-v+1 & u-w+2 & 0 & u-w+1 & u-v+2 \\ w & w-t+1 & w-t+2 & w-u+1 & w-v+2 & 0 & w-u+2 & 0 & w-v+1 \\ v & v-t+1 & v-t+2 & v-u+2 & 0 & v-w+1 & v-u+1 & v-w+2 & 0 \end{bmatrix}$$

The index matrix of the group CS corresponding to this is

$$\mathbf{P} = \begin{bmatrix} w-v & 0 & t & s \\ 0 & w-v & 1 & -u \\ -u & -s & w-v & 0 \\ -1 & t & 0 & w-v \end{bmatrix}$$

with $s = (w-v)^2-ut-1$, elements belonging to $GF(3)$.

Consider the matrix P' in [8] obtained from P by interchanging the digits 1 and 2. Then

$$P' = JP = PJ, \qquad \mathbf{P}' = 2\mathbf{P}.$$

(The reduced form of P' in [4] is, of course, P^*.) That is, P and P' induce identical collineations in Π^{00}_{00} (cf. Theorem 10).

Similar, though much less complicated, forms can be obtained from the matrices representing the 18 projections of the type $(.;1-\epsilon^a\,..)$.

Thus, for the [4], Π^{00}_{00}, there are 45 involution matrices representing the operations of projection. On the basis of these, it is simple to verify the fundamental relations connecting the various projections given in Todd (**13**).

In [8] there are consequently $45 \times 2 = 90$ involutions of the type P, P' belonging to the group CT. From the fact that $P^2 = I$, we deduce immediately that $(XPX^{-1})^2 = I$, for all X.

Choosing $t = 1$, $u = 0$, $v = 1$, $w = 1$, in the matrix P^*, so that it refers to the projection $p(-2;\epsilon\,1\,\epsilon\,\epsilon)$, we find that P, P' (as similarity transform matrices of CT) transform the basic Clifford set into the Clifford sets

$$W^{00}_{02},\ W^{01}_{00},\ W^{11}_{11},\ W^{01}_{11},\ W^{21}_{01}$$

and $\quad W^{00}_{01},\ W^{02}_{00},\ W^{22}_{22},\ W^{02}_{22},\ W^{12}_{02}\quad$ (of course, since $P' = JP$)

respectively.

Theorem 11 proved that 45 Jordan primes in Π^{00}_{00} not intersecting any of the 81 solids Σ lie on the locus L. Other sets of 45 Jordan primes on L will clearly exist when further four-dimensional Π-sections are made.

Consider Π^{01}_{00}. As coordinates in this [4], take

$$x = x_{00},\quad y_1 = x_{10} = x_{20},\quad y_2 = x_{01} = \epsilon^2 x_{02},\quad y_3 = x_{11} = \epsilon^2 x_{22},$$
$$y_4 = x_{21} = \epsilon^2 x_{12}.$$

On calculation, the section of L by Π^{01}_{00} is found to be given by the same set of equations in x, y_i as occurred in the section by Π^{00}_{00}. Though these forms are identical, different interpretations are, of course, attached to the symbols. For instance, the point $(.\,1\,-1\,..)$ which, for Π^{00}_{00} in the

extended coordinate system, is $(.\,1\,1, -1\,.\,.\,, -1\,.\,.)$ becomes

$$(.\,1\,1 -1\,.\,.\, -\epsilon\,.\,.) \quad \text{for } \Pi^{01}_{00}.$$

To complete the cycle, $(.\,1\,1 -1\,.\,.\, -\epsilon^2\,.\,.)$ is the corresponding point for the section by Π^{02}_{00}. (Obviously, it is collinear with the other two points.)

All the sets of 45 Burkhardt points and 45 Jordan solids on L can be obtained by the method indicated. It can be verified that all the points and solids are distinct. The totality of Burkhardt points on L is thus $45 \times 81 = 3645$, with the same number of Jordan solids.

Correspondingly, there are, in CT, $90 \times 81 = 7290$ involutions (that is, 81 sets of 45 pairs) performing the function of a harmonic inversion with respect to a Burkhardt node and its Jordan prime. Such operations are given by $JWP(JW)^{-1}$. In particular, $JW^{01}_{00}P(JW^{01}_{00})^{-1}$ produces the involution matrix P_1 for the section Π^{02}_{00} corresponding to P for Π^{00}_{00}. Of course, P_1 will have the same effect as P on the basic Clifford set.

9. Order and dimension of L

Eliminate x_{00}, x_{01}, x_{02} from the equations $\phi_2 = 0$, $\phi_3 = 0$, $\phi_4 = 0$ given in § 5. After a little calculation, we have

$$\begin{aligned} \frac{x_{10}^3+x_{11}^3+x_{12}^3}{x_{10}x_{11}x_{12}} &= \frac{x_{20}^3+x_{21}^3+x_{22}^3}{x_{20}x_{21}x_{22}} \\ &= \frac{x_{00}^3+x_{01}^3+x_{02}^3}{x_{00}x_{01}x_{02}}, \quad \text{by symmetry,} \\ &= \frac{\theta}{\phi_1}, \quad \textit{addendo.} \end{aligned}$$

Had we eliminated from the equations for ϕ_1, ϕ_3, ϕ_4 instead, we should have obtained

$$\frac{x_{00}^3+x_{10}^3+x_{20}^3}{x_{00}x_{10}x_{20}} = \frac{x_{01}^3+x_{11}^3+x_{21}^3}{x_{01}x_{11}x_{21}} = \frac{x_{02}^3+x_{12}^3+x_{22}^3}{x_{02}x_{12}x_{22}} = \frac{\theta}{\phi_2},$$

and so on.

Therefore, $\phi_1, \phi_2, \phi_3, \phi_4 = 0$ imply $\theta = 0$: that is, the 5 cubic primals are not linearly independent unless one member from each of the products (in the case of θ/ϕ_1, say) $x_{00}x_{01}x_{02}$, $x_{10}x_{11}x_{12}$, $x_{20}x_{21}x_{22}$ (each being a cubic term of $\phi_1 = 0$) vanishes.

Take $x_{00} = 0$, $x_{10} = 0$, $x_{20} = 0$ as typical. This [5] lies on ϕ_1, ϕ_3, ϕ_4, and cuts ϕ_2 in the locus

$$x_{01}x_{11}x_{21}+x_{02}x_{12}x_{22} = 0.$$

This is a *Perazzo primal* P_4^3 in [5] [Perazzo (17)]. Twelve such Perazzo primals occur, by virtue of the Clebsch pattern.

Now each ϕ_i $(i = 1, 2, 3, 4)$ is of order 3, so that their intersection will have total order $3^4 = 81$. Since each is of dimension 7, being a primal in [8], the dimension of their intersection is at least 4. Otherwise stated, their intersection is the locus M_x^{81} $(x \geqslant 4)$.

As partial intersection locus of the ϕ_i the 12 Perazzo primals form a reducible intersection locus $K_4^{3\times 12} = K_4^{36}$, so that $x = 4$. Rejecting this reducible locus, we deduce that the residual intersection of the ϕ_i is an irreducible locus (L) lying on θ (Theorem 4) of dimension 4 and order $81-36 = 45$. That is, the locus L, invariant under the 51840×81 operations of CT, may be written L_4^{45} (see § 7).

THEOREM 12. *The locus L is of order* 45 *and dimension* 4.

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ON RIGHT-MULTIPLICATION ALGEBRAS

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1. Introduction

LET $\mathbf{V}_n$ be an n-dimensional vector space over a field $\mathbf{F}$, and let $\mathbf{A}$ be a linear algebra over $\mathbf{V}_n$. Thus $\mathbf{A}$ consists of the space $\mathbf{V}_n$ together with multiplication defined in such a way that the product of two vectors is a vector, and the identities

$$x(y+z) = xy+xz, \qquad (x+y)z = xz+yz$$
$$x(\alpha y) = \alpha(xy) = (\alpha x)y$$

are satisfied for all $x, y, z \in \mathbf{V}_n$ and all $\alpha \in \mathbf{F}$. The *right-multiplication algebra* $\mathbf{R}(\mathbf{A})$ of $\mathbf{A}$ is the linear associative algebra generated by the transformations of $\mathbf{V}_n$ into itself of the form $x \to xa$, where a is fixed in a given transformation. Similarly the *left-multiplication algebra* $\mathbf{L}(\mathbf{A})$ of $\mathbf{A}$ is the algebra generated by the transformations $x \to ax$. I shall prove theorems concerned mainly with right-multiplication algebras, but similar results hold for left-multiplication algebras.

Two distinct algebras can have both right- and left-multiplication algebras the same. For example, if e_1, e_2 is a basis for a vector space of two dimensions, then the right-multiplication algebras of the algebras whose multiplication tables are

(i)

	e_1	e_2
e_1	e_1	0
e_2	0	e_1

(ii)

	e_1	e_2
e_1	0	e_1
e_2	e_1	0

both consist of all the transformations linearly dependent on $e_1 \to e_1$, $e_2 \to 0$ and $e_1 \to 0$, $e_2 \to e_1$. Similarly the left-multiplication algebras of (i) and (ii) coincide (being the same as the right-multiplication algebras in this case). But the algebras defined by (i) and (ii) are not even isomorphic.

In this paper, I investigate certain classes of algebras each consisting of algebras which possess the same right-multiplication algebra, paying particular attention to the cases in which the right-multiplication algebra is of nullity n.

It is assumed throughout that $n \geqslant 2$ and that $\mathbf{F}$ contains at least three distinct elements other than zero, i.e. $\mathbf{F}$ is neither $\mathbf{GF}(2)$ nor $\mathbf{GF}(3)$.

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2. The nullity and genus of an algebra

The *nullity* (**5**) of an algebra **A** is the minimum number of linearly independent elements which generate **A**, in the sense that every element of **A** is linearly dependent on the generators and their products. The *genus* of an algebra is the difference between its dimension and its nullity. All linear algebras of genus zero and certain types of linear algebras of genus one have recently been classified (**6**, **7**). Some of the results of these papers are required for the purposes of the present paper, and I shall also use the following lemmas.

LEMMA 1. *If* **A** *is a linear algebra over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A})$ *is of nullity* n, *and if* $u_1,\ldots, u_n$ *is any set of* n *linearly independent elements of* $\mathbf{V}_n$, *then the transformations* $x \to xu_i$ $(i = 1,\ldots, n)$ *are* n *independent generators of* $\mathbf{R}(\mathbf{A})$, *and any transformation* $x \to xy$ *is linearly dependent on these transformations.*

Proof. Since $\mathbf{R}(\mathbf{A})$ is generated by the transformations $x \to xy$ and y is linearly dependent on u_i $(i = 1,\ldots, n)$, then $\mathbf{R}(\mathbf{A})$ is generated by $x \to xu_i$ $(i = 1,\ldots, n)$. But the nullity of $\mathbf{R}(\mathbf{A})$ is n, and so these transformations are linearly independent. That any transformation $x \to xy$ is linearly dependent on $x \to xu_i$ $(i = 1,\ldots, n)$ follows at once from the fact that any element of $\mathbf{V}_n$ is linearly dependent on the u_i.

LEMMA 2. *If* **A** *is an algebra over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A})$ *is of nullity* n, *then* $xy = 0$ *for all* x *implies* $y = 0$.

Proof. If
$$y = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n,$$
and Y, $U_1,\ldots, U_n$ are the transformations $x \to xy$, $x \to xu_1,\ldots, x \to xu_n$ respectively, then
$$Y = \alpha_1 U_1 + \ldots + \alpha_n U_n.$$
By Lemma 1, $U_1,\ldots, U_n$ are linearly independent, so that $Y = 0$ only if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$: that is, only if $y = 0$.

LEMMA 3. *Let* **C** *be a linear algebra of dimension* m *and nullity* n *containing a unit element* I *and defined over a field* **F** *which contains at least* $g+2$ *distinct non-zero elements, where* $g = m-n$ *is the genus of* **C**. *Then* I *is not linearly dependent on any set of* n *independent generators of* **C**.

Proof. Let X be an element of **C**, independent of I, and consider the sequence $\{X^r\}$, where X^r is defined inductively by $X^r = X^{r-1}X$. Since **C** is of nullity n, the elements $X, X^2,\ldots, X^{g+2}$ are linearly dependent; otherwise a set of less than n independent generators would exist. Therefore, for some elements $\alpha_1,\ldots, \alpha_{g+1} \in \mathbf{F}$,
$$X^{g+2} = \sum_{s=1}^{g+1} \alpha_s X^s.$$

Let ρ be an element of **F**. Then the elements

$$I+\rho X,\ (I+\rho X)^2,\dots,\ (I+\rho X)^{g+2}$$

are linearly dependent. If I, X, $X^2,\dots$, X^{g+1} are linearly independent, we have

$$\begin{vmatrix} 1 & \rho & 0 & \cdot & \cdot & \cdot \\ 1 & 2\rho & \rho^2 & \cdot & \cdot & \cdot \\ 1 & 3\rho & 3\rho^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \binom{g+1}{1}\rho & \binom{g+1}{2}\rho^2 & \cdot & \cdot & \cdot \\ 1 & \binom{g+2}{1}\rho+\alpha_1\rho^{g+2} & \binom{g+2}{2}\rho^2+\alpha_2\rho^{g+2} & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Hence, if ρ is non-zero, it satisfies a polynomial equation of degree at most $g+1$. But, by hypothesis, **F** contains at least $g+2$ distinct non-zero elements and therefore it is always possible to choose ρ so that the equation is not satisfied. Hence, I, $X,\dots$, X^{g+1} are linearly dependent. Also X is independent of I, and so the least integer k such that

$$I,\ X,\dots,\ X^{k+1}$$

are linearly dependent satisfies $1 \leqslant k \leqslant g$.

Since I, $X,\dots$, X^{k+1} are linearly dependent and I, $X,\dots$, X^k are linearly independent, we have

$$X^{k+1} = \sum_{s=0}^{k} \beta_s X^s \qquad (X^0 = I)$$

for some elements β_0, $\beta_1,\dots$, β_k in **F**. Therefore the elements

$$I+\rho X,\ (I+\rho X)^2,\dots,\ (I+\rho X)^{k+1}$$

can be expressed in terms of the independent elements I, $X,\dots$, X^k as follows:

$$(I+\rho X)^s = I+\binom{s}{1}\rho X+\dots+\rho^s X^s \quad (s = 1, 2,\dots, k),$$

$$(I+\rho X)^{k+1} = \{1+\beta_0\rho^{k+1}\}I+\left\{\binom{k+1}{1}\rho+\beta_1\rho^{k+1}\right\}X+\dots+$$

$$+\left\{\binom{k+1}{k}\rho^k+\beta_k\rho^{k+1}\right\}X^k.$$

Again using the hypothesis that **F** contains at least $g+2$ distinct non-zero elements, we can show that ρ can be chosen so that the determinant of the coefficients of $I, X,..., X^k$ on the right-hand sides of these equations is non-zero. For such a value of ρ the elements

$$I+\rho X,\ (I+\rho X)^2,...,\ (I+\rho X)^{k+1}$$

are linearly independent, and the $k+1$ linearly independent elements $I, X,..., X^k$ can be expressed linearly in terms of them.

Suppose now that I is linearly dependent on a set of n generators of **C**. Then there is a system of n independent generators one of which is I. Since $n > 1$, there is at least one generator independent of I; let X be such a generator. Then, as we have shown, it is possible to find an element $\rho \in \mathbf{F}$ such that I and X are expressible linearly in terms of $I+\rho X,..., (I+\rho X)^{k+1}$. Therefore the generators I and X can be replaced by $I+\rho X$, and so **C** admits a system of $n-1$ linearly independent generators. But this contradicts the assumption that the nullity of **C** is n. Therefore I cannot be linearly dependent on a set of n generators of **C**.

LEMMA 4. *If* **A** *is an algebra over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A})$ *is of nullity n and admits a unit element, and if* **F** *contains at least* $g+2$ *distinct non-zero elements, where g is the genus of* $\mathbf{R}(\mathbf{A})$, *then* $xy+\alpha x = 0$ $(\alpha \in \mathbf{F})$ *for all x implies that* $y = 0$ *and* $\alpha = 0$.

Proof. If $xy+\alpha x = 0$, then the transformation $x \to xy$ is a scalar multiple of the identity. But $x \to xy$ is linearly dependent on a set of n generators of $\mathbf{R}(\mathbf{A})$, by Lemma 1. Hence, by Lemma 3, $\alpha = 0$; then it follows at once from Lemma 2 that $y = 0$.

3. Algebras whose right-multiplication algebras are of genus zero

THEOREM 1. *Let* **C** *be a linear associative algebra of dimension n and genus zero, consisting of transformations of* $\mathbf{V}_n$ *into itself, multiplication being defined in the usual way. Then, if* **A** *and* $\mathbf{A}^*$ *are two algebras over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A}) = \mathbf{R}(\mathbf{A}^*) = \mathbf{C}$, *there exists an automorphism* τ *of* $\mathbf{V}_n$ *such that, for all x, y in* $\mathbf{V}_n$,

$$x \circ y = x\tau(y) \tag{1}$$

where $x \circ y$ *denotes multiplication in* $\mathbf{A}^*$ *and* xy *denotes multiplication in* **A**.

Proof. Since the genus of **C** is zero and its dimension is n, its nullity is n. Let Y be any element of **C**. Then Y is linearly dependent on any set of generators of **C** and so, by Lemma 1, Y is linearly dependent on

$x \to xu_i\,(i = 1, 2, ..., n)$, where $u_1, ..., u_n$ are linearly independent elements of $\mathbf{V}_n$. Hence there exists an element $y' \in \mathbf{V}_n$ such that Y is the transformation $x \to xy'$. Moreover y' is unique, since, if $x \to xy'$ and $x \to xy'_1$ are the same transformation, then $x \to x(y'-y'_1)$ is the zero transformation, and so, by Lemma 2, $y' = y'_1$. Similarly Y defines a unique element y of $\mathbf{V}_n$ such that Y is the transformation $x \to x \circ y$. Given y', the transformation Y is uniquely defined, and hence y is uniquely defined by y'. Similarly y' is uniquely defined by y. It follows that the transformation τ defined by

$$\tau(y) = y'$$

is a one-one transformation of $\mathbf{V}_n$ onto itself such that

$$x \circ y = x\tau(y).$$

If y and z are any two elements of $\mathbf{V}_n$, and if α, β are any two elements of $\mathbf{F}$, then

$$x \circ (\alpha y+\beta z) = x\tau(\alpha y+\beta z),$$

$$\alpha(x \circ y)+\beta(x \circ z) = \alpha x\tau(y)+\beta x\tau(z).$$

Since multiplication in both $\mathbf{A}$ and $\mathbf{A}^*$ is distributive, it follows that

$$x\{\tau(\alpha y+\beta z)-\alpha\tau(y)-\beta\tau(z)\} = 0,$$

and hence, by Lemma 2,

$$\tau(\alpha y+\beta z) = \alpha\tau(y)+\beta\tau(z).$$

Therefore τ is an automorphism, and so Theorem 1 is proved.

4. A special type of isotopy between linear algebras

An algebra $\mathbf{A}^*$ is said to be *isotopic* to an algebra $\mathbf{A}$ if there exist automorphisms ρ, σ, τ of $\mathbf{V}_n$ such that, for all x, y in $\mathbf{V}_n$,

$$x \circ y = \rho\{\sigma(x)\tau(y)\}, \tag{2}$$

where $x \circ y$ denotes multiplication in $\mathbf{A}^*$ and xy denotes multiplication in $\mathbf{A}$. If $\mathbf{A}^*$ is isotopic to $\mathbf{A}$, it is called an *isotope* of $\mathbf{A}$, and the relation defined by (2) is called an *isotopy* of $\mathbf{A}$ onto $\mathbf{A}^*$. Clearly the relation between algebras of being isotopic is an equivalence relation. It is a more general type of equivalence than isomorphism: the latter is given by (2) with $\sigma = \tau = \rho^{-1}$. The concept of isotopy in the general theory of linear algebras was first studied by Albert (**1**) and Bruck (**3**).

If the automorphism ρ in (2) is the identity, then $\mathbf{A}^*$ is called a *principal isotope* of $\mathbf{A}$; any isotope of $\mathbf{A}$ is isomorphic with a principal isotope. In § 2 the relation (1) between algebras is a special type of principal isotopy, in which σ as well as ρ is the identity. This special type of isotopy has been considered by Albert (**2**) in the cases in which τ is an

involution of $\mathbf{A}$. In particular, these include algebras of matrices in which multiplication is performed row by row: such algebras occur when $\mathbf{A}$ is an ordinary algebra of matrices with the property that $P \in \mathbf{A}$ implies $P' \in \mathbf{A}$, where P' is the transpose of the matrix P, and $\tau(P) = P'$. In the present paper, τ is not necessarily an involution.

The proof of the following theorem, which implies the converse of Theorem 1 in the cases in which $\mathbf{R}(\mathbf{A})$ is of dimension n and genus zero, is trivial.

THEOREM 2. *If* $\mathbf{A}$ *is any linear algebra, and* $\mathbf{A}^*$ *is an isotope of the special type given by* (1), *then* $\mathbf{R}(\mathbf{A}) = \mathbf{R}(\mathbf{A}^*)$.

In general, algebras whose right-multiplication algebras coincide need not be isotopic. For example, the algebras having the multiplication tables

(i)

	e_1	e_2	e_3
e_1	e_2	0	0
e_2	0	e_3	0
e_3	0	0	0

(ii)

	e_1	e_2	e_3
e_1	e_2	0	e_3
e_2	0	e_3	0
e_3	0	0	0

both have, as right-multiplication algebra, the algebra of transformations

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where x_1, x_2, x_3 are the components of a vector x in $\mathbf{V}_3$ with respect to the basis e_1, e_2, e_3. But the algebras (i) and (ii) are not isotopic, for the first contains an element t (the element e_3 in terms of the chosen basis) such that $\mathbf{A}t = 0 = t\mathbf{A}$, and no such element exists in the second; hence no relation of the form (2) is possible.

5. Algebras whose right-multiplication algebras are of genus one and admit a unit element

THEOREM 3. *Let* $\mathbf{C}$ *be a linear associative algebra of transformations of* $\mathbf{V}_n$ *into itself, of dimension* $n+1$, *genus one, and admitting a unit element. Then, if* $\mathbf{A}$ *and* $\mathbf{A}^*$ *are two algebras over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A}) = \mathbf{R}(\mathbf{A}^*) = \mathbf{C}$, *there exists an automorphism* τ *of* $\mathbf{V}_n$ *onto itself and a linear function* $h\colon \mathbf{V}_n \to \mathbf{F}$ *such that*

$$x \circ y = h(y)x + x\tau(y), \tag{3}$$

where $x \circ y$ *denotes multiplication in* $\mathbf{A}^*$ *and* xy *denotes multiplication in* $\mathbf{A}$.

Proof. Since the dimension of $\mathbf{C}$ is $n+1$ and its genus is one, its nullity is n. The condition of Lemma 2 is therefore satisfied. Also, since $\mathbf{F}$ is neither $\mathbf{GF}(2)$ nor $\mathbf{GF}(3)$, the conditions of Lemma 4 are satisfied.

Let Y be the transformation $x \to x \circ y$, where $y\ (\neq 0) \in \mathbf{V}_n$. Then, by Lemmas 2 and 4, Y is neither zero nor a scalar multiple of the unit element I of $\mathbf{C}$. However, by Lemma 3, Y is linearly dependent on I and any set of n linearly independent generators of $\mathbf{C}$. But $\mathbf{C} = \mathbf{R}(\mathbf{A})$, and so, by Lemma 1, a system of generators of $\mathbf{C}$ consists of n independent transformations of the form $x \to xz$. Therefore Y can be expressed as a transformation of the form

$$x \to xy' + \lambda(Y)x$$

for some $y' \in \mathbf{V}_n$ and $\lambda(Y) \in \mathbf{F}$. From Lemma 4, it follows that y' and $\lambda(Y)$ are uniquely defined by Y and therefore by y.

Using the same argument, but with the roles of $\mathbf{R}(\mathbf{A})$ and $\mathbf{R}(\mathbf{A}^*)$ interchanged, we see that any transformation $x \to xy'$, denoted by Y', can be expressed in the form

$$x \to x \circ y + \mu(Y')x$$

for some y in $\mathbf{V}_n$ and $\mu(Y')$ in $\mathbf{F}$, both uniquely defined by y'.

It follows that the transformation τ defined by

$$\tau(y) = y'$$

is a one-one transformation of $\mathbf{V}_n$ onto itself such that

$$x \circ y = x\tau(y) + h(y)x,$$

where $h\colon \mathbf{V}_n \to \mathbf{F}$ is defined by

$$h(y) = \lambda(Y) = -\mu(Y'),$$

Y, Y' being the transformations $x \to x \circ y$ and $x \to x\tau(y)$ respectively.

Let y, z be two elements of $\mathbf{V}_n$ and α, β two elements of $\mathbf{F}$. Then

$$x \circ (\alpha y + \beta z) = h(\alpha y + \beta z)x + x\tau(\alpha y + \beta z),$$

$$\alpha(x \circ y) + \beta(x \circ z) = \alpha h(y)x + \alpha x\tau(y) + \beta h(z)x + \beta x\tau(z).$$

Since multiplication in both $\mathbf{A}$ and $\mathbf{A}^*$ is distributive, we have

$$\{h(\alpha y + \beta z) - \alpha h(y) - \beta h(z)\}x + x\{\tau(\alpha y + \beta z) - \alpha\tau(y) - \beta\tau(z)\} = 0.$$

Hence, by Lemma 4,

$$h(\alpha y + \beta z) = \alpha h(y) + \beta h(z),$$

$$\tau(\alpha y + \beta z) = \alpha\tau(y) + \beta\tau(z),$$

and so h is a linear function and τ is an automorphism.

Thus Theorem 3 is proved. The converse is not true; if $\mathbf{A}$ is an algebra such that $\mathbf{R}(\mathbf{A}) = \mathbf{C}$ and $\mathbf{A}^*$ is given by (3), then $\mathbf{R}(\mathbf{A}^*)$ need not be the same as $\mathbf{C}$. However, it is easily shown that, if $\mathbf{A}$ is any linear algebra over $\mathbf{V}_n$ and $\mathbf{A}^*$ is given by (3) for some automorphism τ and linear

function h, then
$$[\mathbf{R}(\mathbf{A}), I] = [\mathbf{R}(\mathbf{A}^*), I],$$
where $[\mathbf{R}(\mathbf{A}), I]$ denotes the algebra obtained from $\mathbf{R}(\mathbf{A})$ by adjoining the identity I, so that, if I is already an element of $\mathbf{R}(\mathbf{A})$, then
$$[\mathbf{R}(\mathbf{A}), I] = \mathbf{R}(\mathbf{A}).$$

The following theorems give more precise conditions in the cases in which $\mathbf{R}(\mathbf{A})$ is of genus one and contains the identity or is of genus zero.

We consider first the case in which $\mathbf{R}(\mathbf{A})$ is of genus one and contains the identity. By a result of a previous paper (7) there exist linear functions p and q such that multiplication in $\mathbf{R}(\mathbf{A})$ is given by
$$UV = p(V)U + q(U)V - p(V)q(U)I, \tag{4}$$
where I is the identity.

THEOREM 4. *Let* $\mathbf{A}$ *be an algebra over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A})$ *is of dimension* $n+1$, *of genus one, and contains the identity* I. *If* $\mathbf{A}^*$ *is an algebra over* $\mathbf{V}_n$ *given by* (3), *where* τ *is an automorphism and* h *is a linear function, then* $\mathbf{R}(\mathbf{A}^*) = \mathbf{R}(\mathbf{A})$ *unless* $h(y) = -p(Y')$ *or* $h(y) = -q(Y')$, Y' *being the transformation* $x \to x\tau(y)$.

If $p = q$ (*i.e.* $\mathbf{R}(\mathbf{A})$ *is commutative*) *and* $h(y) = -p(Y')$, *then* $\mathbf{R}(\mathbf{A}^*)$ *is the zero algebra of dimension* n. *If* $p \neq q$ *and* $h(y) = -p(Y')$ *or* $-q(Y')$, *then* $\mathbf{R}(\mathbf{A}^*)$ *is an algebra of dimension* n *and genus zero.*

Proof. $\mathbf{R}(\mathbf{A}^*)$ is generated by the transformations
$$x \to x \circ y = h(y)x + x\tau(y),$$
and every transformation $x \to x \circ y$ is of the form $Y' - \mu(Y')I$, where Y' is the transformation $x \to x\tau(y)$ and μ is the linear function over $\mathbf{R}(\mathbf{A})$ defined by
$$\mu(Y') = -h(y). \tag{5}$$

Let U, V be the transformations $x \to xu$ and $x \to xv$ respectively, and let U_1, V_1 be the transformations $U - \mu(U)I$, $V - \mu(V)I$ respectively, i.e. the transformations $x \to x \circ \tau^{-1}(u)$ and $x \to x \circ \tau^{-1}(v)$. Then, from (4),
$$\begin{aligned} U_1V_1 &= \{p(V) - \mu(V)\}U_1 + \{q(U) - \mu(U)\}V_1 - \{p(V) - \mu(V)\}\{q(U) - \mu(U)\}I \\ &= p(V_1)U_1 + q(U_1)V_1 - p(V_1)q(U_1)I. \end{aligned} \tag{6}$$
It follows at once that $\mathbf{R}(\mathbf{A}^*)$ contains the identity, and so is the same as $\mathbf{R}(\mathbf{A})$ unless $p(V_1) = 0$ for all V_1 or $q(U_1) = 0$ for all U_1. Hence
$$\mathbf{R}(\mathbf{A}^*) = \mathbf{R}(\mathbf{A})$$
unless $p(U) = \mu(U)$ for all U or $q(U) = \mu(U)$ for all U. Since μ is given in terms of h by (5), the first part of the theorem is proved.

If $p = q$, then $\mathbf{R}(\mathbf{A})$ is commutative. If $h(y) = -p(Y')$ in this case, then (6) becomes $U_1 V_1 = 0$ for all U_1, V_1 and so $\mathbf{R}(\mathbf{A}^*)$ is the zero algebra.

If $p \neq q$, and $h(y) = -p(Y')$, then (6) becomes

$$U_1 V_1 = q(U_1)V_1,$$

and so [see (**6**)] $\mathbf{R}(\mathbf{A}^*)$ is a non-zero algebra of genus zero. Similarly, if $h(y) = -q(Y')$, then (6) becomes

$$U_1 V_1 = p(V_1)U_1,$$

and again $\mathbf{R}(\mathbf{A}^*)$ is a non-zero algebra of genus zero.

We now consider the case in which $\mathbf{R}(\mathbf{A})$ is of genus zero. By a result of (**6**), multiplication in $\mathbf{R}(\mathbf{A})$ is given by either

$$YZ = p(Z)Y \tag{7}$$

or

$$YZ = q(Y)Z, \tag{8}$$

where p and q are linear functions. We consider only the first of these possibilities; a similar result can be obtained in the other case.

THEOREM 5. *Let* $\mathbf{A}$ *be a linear algebra over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A})$ *is of dimension n and genus zero, multiplication in* $\mathbf{R}(\mathbf{A})$ *being given by* (7) *for some function p. Let* $\mathbf{A}^*$ *be an algebra over* $\mathbf{V}_n$ *given by* (3), *where* τ *is an automorphism and h is a linear function.* (i) *If* $h(y) = 0$ *for all y, then* $\mathbf{R}(\mathbf{A}^*) = \mathbf{R}(\mathbf{A})$; (ii) *if* $h(y) = -p(Y')$ *for all y, where* Y' *is the transformation* $x \to x\tau(y)$, *then multiplication in* $\mathbf{R}(\mathbf{A}^*)$ *is given by* (8) *for some function q*; (iii) *for all other functions h,* $\mathbf{R}(\mathbf{A}^*) = [\mathbf{R}(\mathbf{A}), I]$.

Proof. Every transformation $x \to x \circ y$ is of the form $Y' - \mu(Y')I$, where Y' is the transformation $x \to x\tau(y)$ and μ is defined by (5). Multiplication of two such transformations

$$Y_1 = Y' - \mu(Y')I, \qquad Z_1 = Z' - \mu(Z')I$$

is given by

$$Y_1 Z_1 = \{p(Z') - \mu(Z')\}Y_1 - \mu(Y')Z_1 + \mu(Y')\{p(Z') - \mu(Z')\}I.$$

Hence, unless μ or $p - \mu$ is identically zero, $\mathbf{R}(\mathbf{A}^*)$ is an algebra of genus one admitting a unit element; it is, in fact, the algebra obtained from $\mathbf{R}(\mathbf{A})$ by adjoining a unit element. If $\mu = 0$, then $\mathbf{R}(\mathbf{A}^*) = \mathbf{R}(\mathbf{A})$. If $\mu = p$, then multiplication in $\mathbf{R}(\mathbf{A}^*)$ is given by $Y_1 Z_1 = -p(Y')Z_1$.

A special case of this theorem occurs when $p = 0$, i.e. $\mathbf{R}(\mathbf{A})$ is a zero algebra. In this case every algebra $\mathbf{A}^*$ given by (3) with h not identically zero has a commutative right-multiplication algebra of genus one admitting a unit element.

6. Particular cases

Let f be a linear function, not identically zero, over $\mathbf{V}_n$. The set of all transformations of $\mathbf{V}_n$ into itself of the form

$$x \to f(x)z, \tag{9}$$

where $z \in \mathbf{V}_n$, determines a linear associative algebra $\mathbf{B}_n(f)$ of dimension n. It is easily shown that $\mathbf{B}_n(f)$ is isomorphic with the algebra $\mathbf{B}_n$ of genus zero in which multiplication is given by

$$e_i e_j = e_i$$

in terms of some basis $e_1, e_2,..., e_n$ [see (**6**)].

THEOREM 6. *Let* **A** *be a linear algebra of dimension n over* $\mathbf{V}_n$ *such that multiplication is given by*

$$xy = f(x)\tau(y) \tag{10}$$

for some automorphism τ *of* $\mathbf{V}_n$. *Then* $\mathbf{R}(\mathbf{A}) = \mathbf{B}_n(f)$.

Conversely, if **A** *is any algebra over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A}) = \mathbf{B}_n(f)$, *then multiplication in* **A** *is given by* (10) *for some automorphism* τ.

Proof. Let $\mathbf{A}_0$ be the algebra over $\mathbf{V}_n$ defined by

$$xy = f(x)y.$$

Then $\mathbf{R}(\mathbf{A}_0) = \mathbf{B}_n(f)$. Therefore, by Theorem 2,

$$\mathbf{R}(\mathbf{A}) = \mathbf{R}(\mathbf{A}_0) = \mathbf{B}_n(f).$$

Conversely, let **A** be any algebra over $\mathbf{V}_n$ such that $\mathbf{R}(\mathbf{A}) = \mathbf{B}_n(f)$. Then $\mathbf{R}(\mathbf{A}) = \mathbf{R}(\mathbf{A}_0)$ and so, by Theorem 1, **A** is given by (10) for some automorphism τ.

The set of all transformations of $\mathbf{V}_n$ into itself of the form

$$x \to -f(z)x+f(x)z \tag{11}$$

determines a linear associative algebra $\mathbf{C}_n(f)$ of dimension n, isomorphic with the algebra $\mathbf{C}_n$ of genus zero given by

$$e_i e_j = e_j \quad (i, j = 1, 2,..., n).$$

[$\mathbf{C}_n$ is not isomorphic with $\mathbf{B}_n$, but is a transpose of it; see Bruck (**3**) and Etherington (**4**).]

The following theorem can be proved by a method analogous to that used in the proof of Theorem 6.

THEOREM 7. *Let* **A** *be a linear algebra in which multiplication is given by*

$$xy = -f_2(y)x+f(x)\tau(y), \tag{12}$$

where $f_2(y) = f\{\tau(y)\}$ *and* τ *is an automorphism of* $\mathbf{V}_n$. *Then* $\mathbf{R}(\mathbf{A}) = \mathbf{C}_n(f)$.

Conversely, if **A** *is any algebra over* $\mathbf{V}_n$ *such that* $\mathbf{R}(\mathbf{A}) = \mathbf{C}_n(f)$, *then multiplication in* **A** *is given by* (12) *for some automorphism* τ.

The set of all transformations of $\mathbf{V}_n$ into itself of the form

$$x \to \Gamma(x) = \lambda x + f(x)z,$$

where $\lambda \in \mathbf{F}$ and $z \in \mathbf{V}_n$, determines a linear associative algebra $\mathbf{H}_{n+1}(f)$ of dimension $n+1$. If Γ, Γ' are the transformations corresponding to λ, z and λ', z' respectively, then multiplication in $\mathbf{H}_{n+1}(f)$ is given by

$$\Gamma\Gamma' = \lambda\Gamma' + \{\lambda' + f(z')\}\Gamma - \lambda\{\lambda' + f(z')\}I,$$

where I is the identity of $\mathbf{H}_{n+1}(f)$, given by $\lambda = 1$ and $z = 0$. It was shown in a previous paper (**7**) that the algebras $\mathbf{H}_{n+1}(f)$ are all isomorphic with a fixed linear associative algebra $\mathbf{H}_{n+1}$ of genus one; also that, if $\mathbf{A}$ is a linear algebra of genus zero other than a Lie algebra or an associative algebra, then $\mathbf{R}(\mathbf{A})$ and $\mathbf{L}(\mathbf{A})$ are both isomorphic with $\mathbf{H}_{n+1}$. So far as $\mathbf{R}(\mathbf{A})$ is concerned, the latter result is generalized in the following theorem.

THEOREM 8. *Let* $\mathbf{A}$ *be a linear algebra over* $\mathbf{V}_n$ *in which multiplication is given by*

$$xy = g(y)x + f(x)\tau(y), \tag{13}$$

where g *is a linear function over* $\mathbf{V}_n$, *not identically zero, and* τ *is an automorphism of* $\mathbf{V}_n$ *such that* $f_2 + g$ *is not identically zero, where* $f_2(x) = f\{\tau(x)\}$. *Then* $\mathbf{R}(\mathbf{A}) = \mathbf{H}_{n+1}(f)$.

Conversely, if $\mathbf{A}$ *is any algebra such that* $\mathbf{R}(\mathbf{A}) = \mathbf{H}_{n+1}(f)$, *then multiplication in* $\mathbf{A}$ *is given by* (13), *where* τ *is an automorphism and* g *is a linear function such that neither* g *nor* $f_2 + g$ *is identically zero.*

Proof. Let $\mathbf{A}_0$ be the algebra over $\mathbf{V}_n$ defined by

$$xy = f(x)y.$$

Then $\mathbf{A}_0$ is of genus zero and $\mathbf{R}(\mathbf{A}_0)$ is the algebra of dimension n and genus zero in which multiplication is given by

$$UV = p(V)U,$$

where U and V are the transformations $x \to xu$ and $x \to xv$ respectively and $p(V) = f(v)$.

It is easily shown that $\mathbf{H}_{n+1}(f)$ is the algebra obtained from $\mathbf{R}(\mathbf{A}_0)$ by adjoining a unit element. Hence, by Theorem 5, any algebra $\mathbf{A}$ in which multiplication is given by (13) is such that $\mathbf{R}(\mathbf{A}) = \mathbf{H}_{n+1}(f)$ unless (i) g is identically zero or (ii) $g(y) = -p(Y')$, where Y' is the transformation $x \to x\tau(y)$, i.e. $g(y) = -f\{\tau(y)\}$. In these cases, $\mathbf{R}(\mathbf{A})$ is an algebra of genus zero, as shown by Theorems 6 and 7. Thus the first part of Theorem 8 is proved.

Suppose now that $\mathbf{A}^*$ is an algebra given by (13) with neither g nor f_2+g identically zero. Then $\mathbf{R}(\mathbf{A}^*) = \mathbf{H}_{n+1}(f)$, as has already been proved. Suppose that $\mathbf{R}(\mathbf{A}) = \mathbf{H}_{n+1}(f)$. Then $\mathbf{R}(\mathbf{A}) = \mathbf{R}(\mathbf{A}^*)$ and so, by Theorem 3, there is an automorphism ω of $\mathbf{V}_n$ onto itself such that

$$g(y)x+f(x)\tau(y) = h(y)x+x\omega(y),$$

and hence multiplication in $\mathbf{A}$ is given by

$$xy = g_2(y)x+f(x)\tau_2(y),$$

where

$$g_2(y) = g\{\omega^{-1}(y)\}-h\{\omega^{-1}(y)\}, \qquad \tau_2(y) = \tau\{\omega^{-1}(y)\}.$$

Therefore multiplication in $\mathbf{A}$ is given by an equation of the form (13); neither g_2 nor $f(\tau_2)+g_2$ is identically zero since this would imply that $\mathbf{R}(\mathbf{A}) \neq \mathbf{H}_{n+1}(f)$, as shown in the first part of the proof. Thus the theorem is proved.

We observe that, in an algebra defined by (13), the condition $f_2+g = 0$ is satisfied if and only if $f(xy) = 0$ for all x, y. For an algebra satisfying this condition, we have

$$(xy)(zt) = -f_2(zt)xy,$$

and so the derived algebra is an associative algebra of genus zero. In all other cases, an algebra given by (13) coincides with its own derived algebra. The class of algebras given by (13) consists, in fact, of all special isotopes, of the form given by (1), of algebras of genus zero.

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SOME REMARKS ON QUASI-HAUSDORFF TRANSFORMATIONS

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[Received 1 November 1956]

1. CORRESPONDING to any (complex) sequence $\{\mu_n\}$, the Hausdorff transformation† (H,μ_n) is given by

$$u_n = \sum_{m=0}^{n} \binom{n}{m}(\Delta^{n-m}\mu_m)s_m,$$

where we use the notation

$$\Delta\mu_m = \mu_m - \mu_{m+1}, \qquad \Delta^0\mu_m = \mu_m, \qquad \Delta^k\mu_m = \Delta(\Delta^{k-1}\mu_m).$$

Corresponding to any sequence $\{\nu_n\}$, the quasi-Hausdorff transformation (H^*,ν_n) is given by

$$t_n = \sum_{m=n}^{\infty} \binom{m}{n}(\Delta^{m-n}\nu_n)s_m.$$

Some recent work by M. S. Ramanujan‡ has shown that there is a close connexion between the sequence-to-sequence transformations (H,μ_n) and (H^*,μ_{n+1}). In correspondence with me, he has raised the following question. Suppose that (H,μ_n) is regular; it follows from Ramanujan's results that (H^*,μ_{n+1}) is then also regular. Is it then necessarily true that (H,μ_n) and (H^*,μ_{n+1}) are equivalent?§ The main object of this paper is to show that (H,μ_n) need not necessarily imply (H^*,μ_{n+1}) and that (H^*,μ_{n+1}) need not necessarily imply (H,μ_n); some related matters are, however, dealt with.

In order to show that (H,μ_n) need not imply (H^*,μ_{n+1}), it is enough to consider the case in which

$$\mu_n = \left[\binom{n+r}{r}\right]^{-1} = r\int_0^1 t^n(1-t)^{r-1}\,dt,$$

† For the fundamental properties of the Hausdorff and quasi-Hausdorff transformations, see, e.g., G. H. Hardy, *Divergent series* (Oxford, 1949), Chapter XI.

‡ M. S. Ramanujan, 'Series-to-series quasi-Hausdorff transformations', *J. Indian Math. Soc.* 17 (1953), 47–53; 'On Hausdorff and quasi-Hausdorff methods of summability', *Quart. J. of Math.* (Oxford) (2) 8 (1957) 197–213. I am indebted to the author for letting me see the manuscript of the second of these papers before publication.

§ Given two sequence-to-sequence transformations A, B, we say that 'A implies B' if any sequence summable-A is summable-B to the same sum. If A implies B and B implies A, we say that 'A and B are equivalent'.

Quart. J. Math. Oxford (2), 8 (1957), 272-8.

where $r > 0$. Then (H, μ_n) is the same† as the Cesàro method (C, r) while (H^*, μ_{n+1}) will be denoted by (C^*, r). Thus, as may easily be verified, (C^*, r) is the sequence-to-sequence transformation from $\{s_m\}$ to $\{t_n\}$, where

$$t_n = r(n+1) \sum_{m=n}^{\infty} \frac{m(m-1)...(m-n+1)}{(m+r+1)(m+r)...(m-n+r)} s_m. \tag{1}$$

We remark that, for any fixed n, the coefficient of s_m in the sum on the right of (1) is asymptotically equivalent to m^{-2} for large m, and it is easy to prove that (1) converges if and only if

$$\sum_{m=1}^{\infty} \frac{s_m}{m^2} \tag{2}$$

does so. Now it is well known that, while the summability (C, r) of $\{s_m\}$ implies that

$$s_m = o(m^r),$$

no more is true. Thus, when $r > 2$, there is a sequence $\{s_m\}$ summable (C, r), but such that

$$s_m \neq o(m^2).$$

Thus (2) certainly diverges, so that the (C^*, r) method is not applicable.

It is worth while showing that the proposition that (H, μ_n) implies (H^*, μ_{n+1}) may still be false even when $\{\mu_n\}$ is such that (H^*, μ_{n+1}) is necessarily applicable for any sequence summable (H, μ_n). Moreover, the relation between (C, r) and (C^*, r) appears to be of some interest in itself. I shall therefore establish

THEOREM 1. *The proposition that (C, r) implies (C^*, r) is false when $0 < r < 2$, $r \neq 1$, but is true when $r = 1$ or $r = 2$.*

We remark that it is easily proved that, when $r = 2$ and, *a fortiori*, when $r < 2$, the summability (C, r) of $\{s_m\}$ implies the convergence of (2), and hence the applicability of (C^*, r).

In the other direction we have

THEOREM 2. *If $r > 0$ (r an integer), then (C^*, r) implies (C, r).*

I do not know whether or not (C^*, r) implies (C, r) when r is not an integer. But the falsity of the general proposition that (H^*, μ_{n+1}) always implies (H, μ_n) can be shown by the example

$$\mu_n = \alpha^n,$$

where $0 < \alpha < 1$. Then (H^*, μ_{n+1}) gives the transformation

$$t_n = \alpha^{n+1} \sum_{m=n}^{\infty} \binom{m}{n} (1-\alpha)^{m-n} s_m, \tag{3}$$

† See, e.g. Hardy, loc. cit., Theorem 200.

called $T_{1-\alpha}$ by Meyer-König.† As Meyer-König points out, this sums the sequence $\{z^m\}$ in the set of points $D_1(\alpha)$ consisting of $z = 1$ together with the region satisfying both the inequalities

$$(1-\alpha)|z| < 1,$$
$$\alpha|z| < |1-(1-\alpha)z|.$$

On the other hand, (H, μ_n) gives the Euler transformation

$$t_n = \sum_{m=0}^{n} \binom{n}{m}(1-\alpha)^{n-m}\alpha^m s_m, \tag{4}$$

which sums $\{z^m\}$ in the set of points $D_2(\alpha)$ consisting of $z = 1$ together with the region

$$|1-\alpha+\alpha z| < 1.$$

It is easily proved that for any given α (with $0 < \alpha < 1$) $D_1(\alpha)$ includes some points which do not lie in $D_2(\alpha)$. Thus (H^*, μ_{n+1}) does not imply (H, μ_n).

It should be remarked that the result that $(C, 1)$ and $(C^*, 1)$ are equivalent (in other words, the case $r = 1$ of Theorems 1 and 2) is known.‡

2. Proof of Theorem 1

Taking any fixed r, with $0 < r \leqslant 2$, we write

$$\binom{n+r}{n}\sigma_n = \sum_{m=0}^{n} \binom{n-m+r-1}{n-m} s_m,$$

so that σ_n is the nth (C, r) mean of $\{s_m\}$. Then

$$s_m = \sum_{p=0}^{m} \binom{m-p-r-1}{m-p}\binom{p+r}{p}\sigma_p,$$

so that (1) becomes

$$t_n = r(n+1) \sum_{m=n}^{\infty} \frac{m(m-1)...(m-n+1)}{(m+r+1)(m+r)...(m-n+r)} \times$$
$$\times \sum_{p=0}^{m} \binom{m-p-r-1}{m-p}\binom{p+r}{p}\sigma_p. \tag{5}$$

Now we are investigating whether (C, r) implies (C^*, r), so that we have to consider only the case in which σ_p tends to a limit as $p \to \infty$; we may further suppose, without loss of generality, that this limit is zero. I show

† W. Meyer-König, 'Untersuchungen über einige verwandte Limitierungsverfahren', *Math. Z.* 52 (1950), 257–304.

‡ G. H. Hardy, 'A theorem concerning summable series', *Proc. Cambridge Phil. Soc.* 20 (1921), 304–7.

that, under these conditions, we may invert the order of summation in (5). We have†

$$t_n = \lim_{N\to\infty} t_{n,N},$$

where

$$t_{n,N} = r(n+1)\sum_{m=n}^{N} \frac{m(m-1)...(m-n+1)}{(m+r+1)(m+r)...(m-n+r)} \times$$
$$\times \sum_{p=0}^{m} \binom{m-p-r-1}{m-p}\binom{p+r}{p}\sigma_p.$$

Inverting the order of summation in this finite sum, we obtain

$$t_{n,N} = r(n+1)\sum_{p=0}^{N}\binom{p+r}{p}\sigma_p \times$$
$$\times \sum_{m=\max(n,p)}^{\infty} \frac{m(m-1)...(m-n+1)}{(m+r+1)(m+r)...(m-n+r)}\binom{m-p-r-1}{m-p} - R_{n,N},$$

where

$$R_{n,N} = r(n+1)\sum_{p=0}^{N}\binom{p+r}{p}\sigma_p \times$$
$$\times \sum_{m=N+1}^{\infty} \frac{m(m-1)...(m-n+1)}{(m+r+1)(m+r)...(m-n+r)}\binom{m-p-r-1}{m-p}.$$

In order to justify the inversion, it is thus enough to show that, for any fixed n, $R_{n,N} \to 0$ as $N \to \infty$. But, taking n as fixed, we have

$$\begin{aligned} R_{n,N} &= \sum_{p=0}^{N} \sigma_p\, O\Big\{(p+1)^r \sum_{m=N+1}^{\infty} m^{-2}(m-p)^{-r-1}\Big\} \\ &= \sum_{p=0}^{N} \sigma_p\, O\{(p+1)^r N^{-2}(N+1-p)^{-r}\} \\ &= o(1), \end{aligned}$$

and the inversion is now justified. We thus obtain

$$t_n = \sum_{p=0}^{\infty} \alpha_{n,p}\sigma_p, \tag{6}$$

where

$$\alpha_{n,p} = r(n+1)\binom{p+r}{p} \times$$
$$\times \sum_{m=\max(n,p)}^{\infty} \frac{m(m-1)...(m-n+1)}{(m+r+1)(m+r)...(m-n+r)}\binom{m-p-r-1}{m-p}. \tag{7}$$

Now consider the case in which $r \neq 1, 2$. Since $\{\sigma_p\}$ may be an arbitrary sequence tending to 0, the result that (C, r) does not imply (C^*, r)

† We know that this limit exists, in virtue of the remark made after the enunciation of Theorem 1.

will follow if we show that
$$\sum_{p=0}^{\infty} |\alpha_{n,p}|$$
is unbounded. It will be enough, *a fortiori*, to show that
$$\sum_{\frac{1}{2}n \leqslant p \leqslant n-2} \alpha_{n,p}$$
is unbounded. Let c denote a (strictly) positive number depending only on r (possibly different at each occurrence), and let ψ denote -1 if $0 < r < 1$ and $+1$ if $1 < r < 2$. We observe that
$$\binom{p+r}{p} > cp^r \quad (\text{all } p \geqslant 0),$$
$$\frac{m(m-1)...(m-n+1)}{(m+r+1)(m+r)...(m-n+r)} = \frac{1}{r(r+1)} \frac{\binom{m-n+r-1}{m-n}}{\binom{m+r+1}{m}}$$
$$> c\,\frac{(m-n+1)^{r-1}}{m^{r+1}} \quad (1 \leqslant n \leqslant m),$$
$$\psi\binom{m-p-r-1}{m-p} > c(m-p)^{-r-1} \quad (0 \leqslant p \leqslant m-2).$$
It therefore follows from (7) that, for sufficiently large n,
$$\psi \sum_{\frac{1}{2}n \leqslant p \leqslant n-2} \alpha_{n,p} > cn \sum_{\frac{1}{2}n \leqslant p \leqslant n-2} p^r \sum_{m=n}^{\infty} \frac{(m-n+1)^{r-1}(m-p)^{-r-1}}{m^{r+1}}$$
$$> cn \sum_{m=n}^{2n} \frac{(m-n+1)^{r-1}}{m^{r+1}} \sum_{\frac{1}{2}n \leqslant p \leqslant n-2} p^r(m-p)^{-r-1}$$
$$> c \sum_{m=n}^{2n} (m-n+1)^{r-1} \sum_{\frac{1}{2}n \leqslant p \leqslant n-2} (m-p)^{-r-1}$$
$$> c \sum_{m=n}^{2n} (m-n+1)^{-1}$$
$$> c \log n.$$
This establishes the required result.

It remains to consider the cases $r = 1$, $r = 2$. In the case† $r = 1$ we deduce from (7) that
$$\alpha_{n,p} = \begin{cases} 0 & (p \leqslant n-2), \\ -\dfrac{n}{n+2} & (p = n-1), \\ \dfrac{2(n+1)}{(p+2)(p+3)} & (p \geqslant n). \end{cases} \tag{8}$$

† As has already been remarked, this case is not new, but it is given for the sake of completeness.

In the case $r = 2$, we deduce from (7) that

$$\alpha_{n,p} = \begin{cases} 0 & (p \leqslant n-3), \\ \dfrac{n(n-1)}{(n+2)(n+3)} & (p = n-2), \\ \dfrac{6(n+1)(p-2n+1)}{(p+3)(p+4)(p+5)} & (p \geqslant n-1). \end{cases} \tag{9}$$

It can easily be deduced from (8) and (9) that in each of these two cases the transformation from $\{\sigma_p\}$ to $\{t_n\}$ given by (6) is regular. Thus (C, r) implies (C^*, r), and the proof of Theorem 1 is completed.

3. Proof of Theorem 2

Let r be a positive integer. We suppose that $\{s_m\}$ is summable (C^*, r) and have to prove that it is summable (C, r) to the same sum. Thus we are given, in particular, that (1) converges for all n; we note that, since r is an integer, (1) can be written in the form

$$\frac{t_n}{n+1} = \frac{1}{r+1}\,\Delta^{-r}\left\{s_n\Big/\binom{n+r+1}{r+1}\right\}$$
$$= \frac{1}{r+1}\sum_{m=n}^{\infty}\left\{\binom{m-n+r-1}{m-n}s_m\Big/\binom{m+r+1}{r+1}\right\}. \tag{10}$$

We then have

$$s_m\Big/\binom{m+r+1}{r+1} = (r+1)\Delta^r\left\{\frac{t_m}{m+1}\right\} = (r+1)\sum_{p=m}^{m+r}(-1)^{p-m}\binom{r}{p-m}\frac{t_p}{p+1}; \tag{11}$$

this can be verified by substituting (10) in the expression on the right of (11). Defining σ_n as before, we deduce that

$$\sigma_n = (r+1)\Big/\binom{n+r}{n}\sum_{m=0}^{n}\binom{n-m+r-1}{n-m}\binom{m+r+1}{r+1}\times$$
$$\times\sum_{p=m}^{m+r}(-1)^{p-m}\binom{r}{p-m}\frac{t_p}{p+1}$$
$$= \sum_{p=0}^{n+r}\beta_{n,p}\,t_p, \tag{12}$$

where

$$\beta_{n,p} = \frac{(r+1)!\,n!}{(n+r)!\,(p+1)}\times$$
$$\times\sum_{m=\max(0,p-r)}^{\min(n,p)}(-1)^{p-m}\binom{n-m+r-1}{r-1}\binom{m+r+1}{r+1}\binom{r}{p-m}. \tag{13}$$

I shall now show that the transformation from $\{t_p\}$ to $\{\sigma_n\}$ given by

(12) is regular; the theorem will evidently follow. We first note that we may always take the lower limit of summation in (13) as $p-r$, since the extra terms, if any, all vanish. Again,

$$\binom{n-m+r-1}{r-1} = \frac{(n-m+r-1)(n-m+r-2)...(n-m+1)}{(r-1)!},$$

which vanishes for $m = n+1,\ n+2,...,\ n+r-1$. Thus, if we take the upper limit of summation in (13) as p, the extra terms, if any, all vanish except in the case $p = n+r$. Thus

$$\beta_{n,p} = \frac{n!}{(n+r)!\,(p+1)}\left\{\frac{1}{(r-1)!}\gamma_{n,p}+\delta_{n,p}\right\},$$

where

$$\gamma_{n,p} = \sum_{m=p-r}^{p}(-1)^{p-m}\binom{r}{p-m}(n-m+r-1)\times$$
$$\times(n-m+r-2)...(n-m+1)(m+r+1)(m+r)...(m+1)$$
$$= (-1)^r[\Delta^r\{(n-m+r-1)(n-m+r-2)...(n-m+1)\times$$
$$\times(m+r+1)(m+r)...(m+1)\}]_{m=p-r}, \qquad (14)$$

where the operator Δ^r applies to the variable m and where

$$\delta_{n,p} = \begin{cases}(-1)^r(n+2r+1)(n+2r)...(n+r+1) & (p = n+r),\\ 0 & (\text{otherwise}).\end{cases}$$

Using the result that (under the appropriate conditions)

$$|\Delta^r f(m)| \leqslant \max_{m\leqslant x\leqslant m+r}\left|\frac{d^r f(x)}{dx^r}\right|,$$

we deduce from (14) that for large n, uniformly in p for $0 \leqslant p \leqslant n+r$,

$$\gamma_{n,p} = O\{n^{r-1}(p+1)\},$$

and hence that

$$\beta_{n,p} = O(n^{-1}) \quad (p \neq n+r), \qquad (15)$$

$$\beta_{n,n+r} = O(1). \qquad (16)$$

It clearly follows from (15) and (16) that

$$\sum_{p=0}^{n+r}|\beta_{n,p}|$$

is bounded, and that, for any fixed p,

$$\beta_{n,p} \to 0$$

as $n \to \infty$. Further, if we take $s_m = 1$ for all m, then $\sigma_n = 1$ for all n, and $t_p = 1$ for all p. Hence

$$\sum_{p=0}^{n+r}\beta_{n,p} = 1,$$

and the proof of the regularity of (12) is thus completed.

NOTE ON AN INTEGRABILITY THEOREM FOR SINE SERIES

By SIOBHAN O'SHEA (*Cork*)

[Received 17 December 1956]

THIS note concerns series of the form

$$\sum_{n=1}^{\infty} b_n \sin nx \quad (b_n \geqslant 0). \tag{1}$$

It is well known that, if $\qquad b_n \geqslant b_{n+1} \to 0, \qquad$ (2)

then (1) converges uniformly in every closed subinterval of the open interval $(0, 2\pi)$, and its sum-function $g(x)$ is then continuous in the latter interval. The integrability properties of $g(x)$ in a right-hand neighbourhood of $x = 0$ have been considered by Young (**3**), Boas (**1**), and Heywood (**2**). The following theorem was proved for $\gamma = 0$ by Young and extended to $0 < \gamma \leqslant 1$ by Boas: *if* $0 \leqslant \gamma \leqslant 1$ *and if* (2) *holds, then*

$$x^{-\gamma} g(x) \in L(0, \pi) \tag{3}$$

if and only if

$$\sum_{1}^{\infty} n^{\gamma-1} b_n < \infty. \tag{4}$$

Heywood proved that *this theorem is true also for* $1 < \gamma < 2$.

In this note I show how Heywood's result can be obtained even without assuming (2). I prove the following theorem.

THEOREM. *Suppose that* $1 < \gamma < 2$. *Then* (1) *converges everywhere, to a function* $g(x)$ *satisfying* (3), *if and only if* (4) *is true.*

Heywood has already proved the 'if' part of this theorem, since he makes no use of (2) in his proof of the 'if' part of Theorem 1 of (**2**). It therefore suffices to prove the 'only if' part of the theorem. As a matter of fact, this proof can be reversed, but the proof of the 'if' statement, obtained in this way, is less concise than that of Heywood.

Suppose then that (1) converges everywhere to a function $g(x)$ satisfying (3). We require the following result, due to Boas [(**1**) 219, 220].

LEMMA. *If* $\theta_n \geqslant 0$, *then*

$$x^{-1} \sum_{1}^{\infty} \theta_n \sin nx \in L(0, \pi)$$

implies

$$\sum_{1}^{\infty} \theta_n < \infty,$$

and, for $0 < \alpha < 1$,

$$x^{-\alpha} \sum_{0}^{\infty} \theta_n \cos nx \in L(0, \pi)$$

implies

$$\sum_{1}^{\infty} n^{\alpha-1}\theta_n < \infty.$$

It should be observed that $\{\theta_n\}$ need not be monotonic provided that the trigonometric series involved are convergent.

Since $b_n \geqslant 0$, and since $x^{-1}g(x) \in L(0, \pi)$ by (3), it follows from the first part of the lemma, with $\theta_n = b_n$, that

$$\sum_{1}^{\infty} b_n < \infty.$$

We are therefore justified in defining a sequence $\{B_n\}$ by

$$B_n = \sum_{n}^{\infty} b_j,$$

and B_n then decreases steadily to 0 as $n \to \infty$. Hence the series

$$G(x) = B_1 + \sum_{1}^{\infty} (B_n + B_{n+1})\cos nx$$

converges uniformly in every closed subinterval of the open interval $(0, 2\pi)$, so that $G(x)$ is continuous in the latter interval.

There is a very simple relation between $g(x)$ and $G(x)$. Using summation by parts and the trivial identity

$$\cos(n+1)x + \cos nx = \cot \tfrac{1}{2}x\{\sin(n+1)x - \sin nx\},$$

we have

$$\begin{aligned} G(x) &= \sum_{0}^{\infty} B_{n+1}\{\cos(n+1)x + \cos nx\} \\ &= \cot \tfrac{1}{2}x \sum_{0}^{\infty} B_{n+1}\{\sin(n+1)x - \sin nx\} \\ &= \cot \tfrac{1}{2}x \sum_{0}^{\infty} (B_n - B_{n+1})\sin nx \\ &= g(x) \cot \tfrac{1}{2}x. \end{aligned}$$

It follows easily from this and (3) that

$$x^{-\gamma+1}G(x) \in L(0, \pi). \tag{5}$$

The argument is now completed by an appeal to the second part of the lemma with $\alpha = \gamma - 1$ and, for $n \geqslant 1$, $\theta_n = B_n + B_{n+1}$. By (5) this gives

$$\sum_{1}^{\infty} n^{\gamma-2}B_n \leqslant \sum_{1}^{\infty} n^{\gamma-2}(B_n + B_{n+1}) < \infty,$$

and from this we deduce (4) by a standard argument involving summation by parts. This proves the theorem.

I conclude with some remarks on the case $0 \leqslant \gamma \leqslant 1$. If we drop the condition (2) (but retain the condition $b_n \geqslant 0$) then, for $0 \leqslant \gamma \leqslant 1$, (4) is implied by the convergence of (1) everywhere to a function $g(x)$ satisfying (3) [(**1**) 219]. But, even if we assume $b_n \to 0$, (4) does not imply the convergence of (1) in the case $0 \leqslant \gamma < 1$, nor does it imply (3) when $\gamma = 1$. Thus the series

$$\sum_1^\infty j^{-\frac{1}{2}} \sin 2^j x$$

diverges almost everywhere [(**4**) 120] even though (4) holds for every $\gamma < 1$. For $\gamma = 1$ a counterexample is provided by

$$\sum_1^\infty (j!)^{-1} \sin 2^{(j!)^2} x,$$

as may be shown by a method very similar to one used in (**4**) 111. However, it seems worthy of note that the equivalence of (3) and (4) survives for $0 < \gamma \leqslant 1$ if we replace the monotonicity of $\{b_n\}$ by *quasi-monotonicity*, i.e. by the requirement that $\{n^{-k} b_n\}$ be non-increasing for some constant $k \geqslant 0$. In fact the following theorem is true.

Suppose that $0 < \gamma \leqslant 1$, *that* $b_n \to 0$ $(n \to \infty)$, *and that* $\{b_n\}$ *is quasi-monotonic. Then* (1) *converges everywhere, to a function* $g(x)$ *satisfying* (3), *if and only if* (4) *holds.*

I offer no proof of this theorem since Boas's proof of Theorem 1 of (**1**) goes through with very little change if we take account of elementary properties of quasi-monotonic sequences. It does not seem possible to replace the monotonicity of $\{b_n\}$ by quasi-monotonicity in the case $\gamma = 0$.

I wish to thank Professor P. B. Kennedy for suggesting the problem treated in this note.

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A NOTE ON MORAN'S THEORY OF DAMS

By F. DOWNTON (*Exeter*)

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In a recent paper (**4**) Moran has derived the Fourier transform of the distribution in equilibrium of the quantity of material in an infinite store, where the input to and the output from the store take place continuously. He did this by considering a simple single server queue (regarding it as a store, which material enters and leaves discretely) and by letting the intervals of time between the discrete changes, and also the size of these discrete changes, tend to zero. It will be shown here that, by considering the single server queue from a different point of view and taking the limit in a different way, slightly more general results can be obtained.

The method depends upon the fact that the mathematics of a single server queue may be regarded also as representing a semi-continuous storage system in which batches of material of varying size enter the store, and the store is subject to a steady drain. This fact has been noted by Smith (**5**) and further exploited by Takács (**6**).

Consider a single server queue in which the time u between successive arrivals has distribution function given by

$$\mathrm{pr}(u \leqslant t) \equiv A(t) \quad (0 \leqslant t < \infty). \tag{1}$$

Let the customers be served in order of arrival and let the length of service time be v, with distribution function

$$\mathrm{pr}(v \leqslant t) \equiv B(t) \quad (0 \leqslant t < \infty). \tag{2}$$

We assume that all such inter-arrival times and service times are mutually independent and that the queue is in statistical equilibrium. It is well known that this is possible if and only if the traffic intensity ρ is less than unity, where

$$\rho = \int\limits_0^\infty t\,dB(t) \Big/ \int\limits_0^\infty t\,dA(t). \tag{3}$$

We define w, the equilibrium distribution of waiting time,† by the relation

$$\mathrm{pr}(w \leqslant t) \equiv C(t) \quad (0 \leqslant t < \infty), \tag{4}$$

† We follow Smith in defining the time spent in the queue by a customer, including his service time, as the 'waiting time'. The expression 'queueing time' would refer to the time spent in the queue before service.

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and in many cases this distribution, or its Laplace transform, can be found in terms of the arrival interval and service-time distributions.

Consider now an infinite store in which batches of material of size v (with distribution as above) arrive irregularly, such that the interval between successive arrivals is u (with distribution again defined above). Then, if this store is subject to a uniform drain at a unit rate, the distribution of the quantity of material in the store at any time is identical with the distribution of the 'virtual' waiting time for the queue defined above, i.e. the waiting time of a hypothetical customer joining the system at that epoch. To obtain this distribution for the store with input and output both continuous we merely let the batch size tend to zero, at the same time letting the interval between successive arrivals also tend to zero.

We first suppose that batches arrive at random: that is that

$$A(t) = 1-e^{-nt} \quad (0 \leqslant t < \infty), \tag{5}$$

where the average arrival interval is given by $1/n$.

For the present we maintain the generality of the distribution of batch size, merely putting $B(t) = B_n(t)$. For the corresponding waiting-time distribution we put $C(t) = C_n(t)$, where both $B_n(t)$ and $C_n(t)$ depend upon n in some way still to be specified although we note that, from equation (3),

$$\int_0^\infty t\,dB_n(t) = \rho/n.$$

We let the Laplace transform of $dB_n(t)$, which will exist for at least all real, positive values of s, be

$$\beta_n(s) \equiv \int_0^\infty e^{-st}\,dB_n(t). \tag{6}$$

We can then use a result given by Takács (6) [110, equation (19)] for the Laplace transform of the waiting time distribution in equilibrium,

$$\gamma_n(s) \equiv \int_0^\infty e^{-st}\,dC_n(t) = (1-\rho)\left\{1-\frac{n[1-\beta_n(s)]}{s}\right\}^{-1}. \tag{7}$$

The Laplace transform of the quantity of material in a suitable semi-continuous store is identical with this Laplace transform of the 'virtual' waiting-time distribution. We require the function $\psi(s)$ given by considering the behaviour of the system as $n \to \infty$: that is

$$\psi(s) = \lim_{n\to\infty} \gamma_n(s).$$

Then $\psi(s)$ will be the Laplace transform of the distribution of the quantity of material in a fully continuous store provided that, as $n \to \infty$, the distribution of the amount of material entering the store in finite time can be shown to retain some meaning.

Consider now the distribution of the quantity of material arriving during time t for the discrete case. The probability that exactly r batches will arrive during time t is given by $(nt)^r e^{-nt}/r!$, and the distribution of the total amount contained in these r batches is the r-fold convolution of $B_n(t)$.

Thus, if the probability that an amount less than x arrives in time t is $F_n(x,t)$ and we have the Laplace transform $\phi_n(s,t)$ given by

$$\phi_n(s,t) \equiv \int_0^\infty e^{-sx}\, dF_n(x,t), \tag{8}$$

then
$$\phi_n(s,t) = \sum_{r=0}^\infty (nt)^r e^{-nt}[\beta_n(s)]^r/r! = \exp\{-nt[1-\beta_n(s)]\}. \tag{9}$$

We require $F_n(x,t)$ to tend, as n tends to infinity, to some function $F(x,t)$, the distribution function for the continuous input in time t.

Let
$$\phi(s,t) \equiv \int_0^\infty e^{-sx}\, dF(x,t). \tag{10}$$

Then, if we use the continuity theorem referred to in Kendall (**1**), the above requirement is equivalent to

$$\lim_{n\to\infty} n[1-\beta_n(s)] = -t^{-1}\log\phi(s,t). \tag{11}$$

Since the left-hand side of this relation does not contain t, neither does the right-hand side. Thus this limiting process can give rise to a solution of our problem only when the continuous input has an additive distribution. This is, in fact, the only type of input distribution which could produce a meaningful equilibrium solution.

Substituting for $n[1-\beta_n(s)]$ in equation (7) and taking the limit as $n \to \infty$ we have†

$$\psi(s) = (1-\rho)\left\{1+\frac{\log\phi(s,t)}{st}\right\}^{-1}. \tag{12}$$

As a special case, if we put, as Moran did,

$$F(x,t) = \frac{1}{\Gamma(t)\rho^t}\int_0^x e^{-y/\rho} y^{t-1}\, dy, \tag{13}$$

† A result equivalent to this (together with some other results of interest) has been obtained independently by Lindley and Smith (**3**).

then $$\phi(s,t) = (1+\rho s)^{-t},$$

and we obtain for the Laplace transform of the distribution of the quantity of material in the store

$$\psi(s) = \frac{(1-\rho)s}{s-\log(1+\rho s)}, \tag{14}$$

which is equivalent to Moran's equation (8).

It may be noticed that in deriving equation (12) it is not necessary to determine what functional form $\beta_n(s)$ takes. For the case giving rise to equation (14) above it is fairly easy to see that all the requirements are satisfied if

$$\beta_n(s) = (1+\rho s)^{-1/n}: \tag{15}$$

that is, the original batch size had a Pearson type-III distribution, though this distribution is not unique in giving the required result.

Returning to the general case, if we define $g(x)$ and $dH(x)$ as the inverses of the Laplace transforms $-(st)^{-1}\log\phi(s,t)$ and $\psi(s)$, respectively, equation (12) can be written in the form

$$H(x) = (1-\rho)+\int\limits_0^x H(y)g(x-y)\,dy. \tag{16}$$

This integral equation of the familiar Volterra type is similar to that arising in renewal theory, and numerical methods for its solution are available. If $g(x)$ has a convenient form in terms of tabulated functions, such a numerical solution may well be simpler than other methods of obtaining values of $H(x)$. The integral equation itself may be regarded as the limit, under suitable conditions, of the integral equation of queue theory given by Lindley (**2**) [equations (4) and (5)].

The behaviour of $\psi(s)$ and $H(x)$ as ρ tends to unity from below is analogous to the behaviour of the queue equation, from which they have been derived. Because of the interest in the situation resulting when ρ is close to unity, Kendall (**1**) has investigated the limiting behaviour (as $\rho \to 1$) of the distribution function of the random variable $(1-\rho)z$, where z has the distribution function

$$\mathrm{pr}(z \leqslant x) \equiv H(x) \tag{17}$$

such that $$\int\limits_0^\infty e^{-sx}\,dH(x) = \frac{(1-\rho)s}{s-\log(1+\rho s)}. \tag{18}$$

This is the special case discussed by Moran, this transform being given by equation (14) of the present note.

Mr. D. G. Kendall has pointed out to me that his argument can be

generalized, and an outline of this is given below. A rigorous treatment requires the continuity theorem for Laplace transforms already mentioned above.

Let $K(x)$ be the distribution function of $(1-\rho)z$ and let

$$\chi(s) \equiv \int\limits_0^\infty e^{-sx}\,dK(x) = \psi\{(1-\rho)s\}. \tag{19}$$

Thus
$$\chi(s) = (1-\rho)\left\{1+\frac{\log\phi[(1-\rho)s,t]}{(1-\rho)st}\right\}^{-1}. \tag{20}$$

Bearing in mind the remark already made that the right-hand side of this expression is independent of t, and putting the mean and variance of the input to the store *per unit time* equal to ρ and σ^2, respectively, we have

$$\frac{\log\phi[(1-\rho)s,t]}{(1-\rho)st} = -\rho+\tfrac{1}{2}(1-\rho)\sigma^2 s+O[(1-\rho)^2]. \tag{21}$$

It follows from equation (20) that

$$\lim_{\rho\to 1}\chi(s) = [1+\tfrac{1}{2}\sigma^2 s]^{-1}, \tag{22}$$

i.e.
$$\lim_{\rho\to 1}K(x) = 1-e^{-2x/\sigma^2}. \tag{23}$$

In the special case of the Pearson type-III input of equation (13), we have $\sigma^2 = \rho^2$, which, as $\rho \to 1$, gives the result obtained by Kendall for this limiting distribution.

I am indebted to Mr. D. G. Kendall for his helpful suggestions and to the referee for his constructive comments.

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INEQUALITIES FOR THE BESSEL FUNCTION $I_n(x)$

By P. A. P. MORAN (*Canberra*)

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IN this note we consider a class of inequalities which hold for the Bessel function

$$I_n(x) = (2\pi)^{-1} \int\limits_0^{2\pi} \exp(x\cos\theta)\cos n\theta\, d\theta$$

when x is real and positive and n is an integer. Consider first the case $n = 0$. Then $I_0(x)$ is the average value of $\exp(x\cos\theta)$ on the unit circle. If we replace the integral by an approximating sum over m points equally spaced around the circle, we get

$$S(\delta) = m^{-1} \sum_{s=1}^{m} \exp\{x\cos(\delta+2\pi s m^{-1})\},$$

where $0 \leqslant \delta \leqslant 2\pi m^{-1}$. For simplicity we restrict ourselves to the case in which m is even. $S(\delta)$ is a periodic function of δ with period $2\pi m^{-1}$. I shall show that

$$S(\pi m^{-1}) \leqslant I_0(x) \leqslant S(0), \tag{1}$$

which may be written

$$I_0(x) \geqslant 2m^{-1}\{\cosh(x\cos\pi m^{-1})+\cosh(x\cos 3\pi m^{-1})+ +\ldots+\cosh(x\cos(m-1)\pi m^{-1})\}$$

and

$$I_0(x) \leqslant 2m^{-1}\{\cosh x+\cosh(x\cos 2\pi m^{-1})+\ldots+\cosh(x\cos(m-2)\pi m^{-1})\}.$$

These inequalities are remarkably accurate and the accuracy increases very rapidly when m is increased. Thus for $I_0(1) = 1{\cdot}26606588$ and $m = 6$ we have

$$\tfrac{1}{3}\{\cosh 1+2\cosh\tfrac{1}{2}\} = 1{\cdot}26611085,$$

$$\tfrac{1}{3}\{2\cosh(\tfrac{1}{2}\surd 3)+1\} = 1{\cdot}26602090,$$

whilst the mean of these two numbers is $1{\cdot}26606588$, which is itself an upper bound for $I_0(1)$ (with $m = 12$), as can be verified by using more decimal places. When x becomes larger, the bounds become less close when m is kept fixed. Thus for $I_0(4) = 11{\cdot}301922$ we get

$$\tfrac{1}{3}\{\cosh 4+2\cosh 2\} = 11{\cdot}610875,$$

$$\tfrac{1}{3}\{2\cosh(2\surd 3)+1\} = 10{\cdot}993020$$

with a mean $11{\cdot}301947$, which is itself an upper bound.

To prove these inequalities we first observe that $\exp(x\cos\theta)$ can be expanded in a series of powers of $\cos\theta$ with coefficients which are all positive when x is positive. It is therefore sufficient to prove that

$$S_1(\pi m^{-1}) \leqslant (2\pi)^{-1}\int\limits_0^{2\pi} \cos^p\theta\, d\theta \leqslant S_1(0), \tag{2}$$

where p is a positive integer and

$$S_1(\delta) = m^{-1}\sum_{s=1}^{m} \cos^p(\delta+2\pi s m^{-1}).$$

The sum and the integral are both zero when p is odd. For p even, $S(\delta)$ is an even periodic function of δ in the interval $0 \leqslant \delta \leqslant 2\pi m^{-1}$, and we expand it in a Fourier cosine series [cf. Titchmarsh (1) Ch. XIII],

$$S_1(\delta) = A_0+A_1\cos m\delta+A_2\cos 2m\delta+\dots,$$

where

$$A_0 = (2\pi)^{-1}\int\limits_0^{2\pi} \cos^p\theta\, d\theta,$$

$$A_r = 0 \quad (p \text{ odd; or } p \text{ even, } rm > p),$$

$$A_r = \frac{2^{-p}p!}{\{\frac{1}{2}(p+rm)\}!\{\frac{1}{2}(p-rm)\}!} \quad (p \text{ even, } rm \leqslant p).$$

Since these coefficients are all positive, it follows that

$$S_1(\delta) \leqslant A_0+A_1+\dots = S_1(0).$$

This proves the right-hand inequality in (2). On the other hand we have

$$A_1 \geqslant A_2 \geqslant A_3 \geqslant \dots,$$

and so
$$S_1(\pi m^{-1}) = A_0-A_1+A_2-A_3+\dots \leqslant A_0,$$

which proves the left-hand inequality in (2), and (1) is therefore proved.

Now consider

$$I_n(x) = (2\pi)^{-1}\int\limits_0^{2\pi} \exp(x\cos\theta)\cos n\theta\, d\theta. \tag{3}$$

As a numerical example take $I_3(3) = 0{\cdot}959754$. For $m = 12$ we have an upper bound

$$\tfrac{1}{6}\{\sinh 3-2\sinh 1{\cdot}5\} = 0{\cdot}959886$$

and a lower bound

$$\tfrac{1}{3}\{\sinh(3\cos(\tfrac{1}{12}\pi))\cos\tfrac{1}{4}\pi+\sinh(3\cos\tfrac{1}{4}\pi)\cos\tfrac{3}{4}\pi+\sinh(3\cos(\tfrac{5}{12}\pi))\cos\tfrac{5}{4}\pi\}$$
$$= 0{\cdot}959622.$$

The mean of these two bounds is $0{\cdot}959754$ to six decimal places.

Thus we are led to the conjecture that the following inequalities exist for $x > 0$, n a positive even integer, and m even:

$$I_n(x) \leqslant 2m^{-1}[\cosh x+\cosh\{x\cos(2\pi/m)\}\cos(2n\pi/m)+$$
$$+\ldots+\cosh\{x\cos((m-2)\pi/m)\}\cos((m-2)n\pi/m)],$$
$$I_n(x) \geqslant 2m^{-1}[\cosh\{x\cos(\pi/m)\}\cos(n\pi/m)+\cosh\{x\cos(3\pi/m)\}\cos(3n\pi/m)+$$
$$+\ldots+\cosh\{x\cos((m-1)\pi/m)\}\cos(n(m-1)\pi/m)],$$

whilst, if n is odd and positive,

$$I_n(x) \leqslant 2m^{-1}[\sinh x+\sinh\{x\cos(2\pi/m)\}\cos(2n\pi/m)+$$
$$+\ldots+\sinh\{x\cos((m-2)\pi/m)\}\cos\{(m-2)n\pi/m\},$$
$$I_n(x) \geqslant 2m^{-1}[\sinh\{x\cos(\pi/m)\}\cos(n\pi/m)+\sinh\{x\cos(3\pi/m)\}\cos(3n\pi/m)+$$
$$+\ldots+\sinh\{x\cos((m-1)\pi/m)\}\cos\{n(m-1)\pi/m].$$

It is sufficient, as before, to consider inequalities between

$$(2\pi)^{-1}\int\limits_0^{2\pi}\cos^p\theta\cos n\theta\,d\theta$$

and
$$S_2(\delta) = m^{-1}\sum_{s=1}^{m}\cos^p(\delta+2\pi sm^{-1})\cos(n\delta+2\pi nsm^{-1}).$$

I shall prove only the result for $m \geqslant n$. This is the only interesting case numerically since, to get a reasonably close estimate, m must be large enough for the sum to take account of the oscillating term $\cos n\theta$. Expanding $S_2(\delta)$ in a Fourier cosine series, we get

$$S_2(\delta) = A_0+A_1\cos m\delta+A_2\cos 2m\delta+\ldots,$$

where

$$A_r = (4\pi)^{-1}\int\limits_0^{2\pi}\cos^p\theta\{\cos(n+rm)\theta+\cos(n-rm)\theta\}\,d\theta,$$

$$A_r = 0 \quad (p \text{ even, } n \text{ odd}),$$

$$A_r = 0 \quad (p \text{ odd, } n \text{ even}),$$

$$A_r = 2^{-p-1}\left\{\frac{p!}{\{\frac{1}{2}(p+n+rm)\}!\{\frac{1}{2}(p-n-rm)\}!}+\right.$$
$$\left.+\frac{p!}{\{\frac{1}{2}(p+n-rm)\}!\{\frac{1}{2}(p-n+rm)\}!}\right\}$$
$$(p,\ n \text{ both even or odd}).$$

Then $A_r \geqslant 0$ and $A_r \geqslant A_{r+1}$ so long as $m > n$, $r \geqslant 1$.

The previous argument then holds and the inequalities are proved. It is possible to estimate the accuracy of these inequalities, for the sum corresponding to (3), namely

$$S_3(\delta) = m^{-1}\sum_{s=1}^{m}\exp(x\cos(\delta+2\pi s m^{-1}))\cos(n\delta+2\pi n s m^{-1}),$$

can be expanded in the Fourier cosine series

$$I_n(x)+\tfrac{1}{2}\{I_{n+m}(x)+I_{n-m}(x)\}\cos m\delta+\tfrac{1}{2}\{I_{n+2m}(x)+I_{n-2m}(x)\}\cos 2m\delta+\dots,$$

and so the difference between the upper and lower bounds is

$$I_{n+m}(x)+I_{n+3m}(x)+I_{n+5m}(x)+\dots+I_{n-m}(x)+I_{n-3m}(x)+I_{n-5m}(x)+\dots.$$

This method of approximation can be applied to any function defined by a formula of the type

$$f(x) = (2\pi)^{-1}\int_0^{2\pi} F(x,\theta)\,d\theta,$$

and the reason why it works so well is that the leading term in the error is the mth Fourier coefficient of $F(x,\theta)$, which decreases exponentially with m when $F(x,\theta)$ is analytic in θ. If, in addition, $F(x,\theta)$ is an even function of θ, it is often possible to establish inequalities of the above kind, but each case has to be considered separately.

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A NOTE ON CHARACTERISTIC VALUES OF PRODUCTS OF TWO MATRICES

By C. R. MARATHE (*Aligarh, India*)

[Received 15 February 1957]

ALL the matrices that we consider have complex elements. P^* will denote the conjugate transpose of the matrix P. For any matrix P, P^*P is positive, i.e. at least positive semi-definite. Further, if $r(P)$ denotes the rank of P, then $r(P^*P) = r(P)$. Let $c(Q)$ stand for *any* characteristic value of the (square) matrix Q. Let $c_{\max}(P^*P)$ and $c_{\min}(P^*P)$ denote respectively the maximum and the minimum characteristic value of P^*P.

THEOREM 1.

(i) $c((AB)^*AB) \leqslant c_{\max}(A^*A)c_{\max}(B^*B)$ *when* A *is an* $n\times m$ *matrix and* B *an* $m\times p$ *matrix*;

(ii) $c((AB)^*AB) \geqslant c_{\min}(A^*A)c_{\min}(B^*B)$ *when* A *is an* $n\times m$ *matrix and* B *an* $m\times n$ *matrix.*

Proof. (i) For a column vector $x = (x_i)$ define

$$\|x^*\| = \|x\| = \Big(\sum_i |x_i|^2\Big)^{\frac{1}{2}}.$$

For column vectors x and y and a *rectangular* matrix A define the *modulus* $\|A\|$ of A by

$$\|A\| = \sup\{|x^*Ay|\}, \qquad \|x^*\| = \|y\| = 1. \tag{1}$$

The modulus of A, so defined, satisfies the usual *norm* conditions [*see* (**5**)]. Further,

$$\|A\| = \|A^*\|, \qquad \|A^*A\| = \|AA^*\|, \qquad \|A^*A\| = \|A\|^2 = c_{\max}(A^*A).$$

Hence

$$\|(AB)^*AB\| = \|AB\|^2 \leqslant \|A\|^2\|B\|^2 = \|A^*A\|\,\|B^*B\|.$$

Since $\|P^*P\| = c_{\max}(P^*P)$, we get (i).

(ii) When at least one of A^*A and B^*B is singular, the result holds (even when B is an $m\times p$ matrix) since $(AB)^*AB$ is positive. Hence, we need consider only the case when *both* A^*A and B^*B are non-singular. Now A^*A is of order $m\times m$ and B^*B of order $n\times n$. Also $r(A^*A) = r(A)$, $r(B^*B) = r(B)$, so that $r(A) = m$ and $r(B) = n$. Hence $n \geqslant m$ and also $m \geqslant n$. Thus $m = n$ and A, B, AB are all n-square non-singular

matrices. Take A^{-1} in place of A^*, and B^{-1} in place of B^* in (i). This gives

$$c(\{(AB)^*AB\}^{-1}) \leqslant c_{\max}(A^*A)^{-1}c_{\max}(B^*B)^{-1}.$$

But characteristic values of P and P^{-1} are reciprocal. Hence

$$c((AB)^*AB) \geqslant c_{\min}(A^*A)c_{\min}(B^*B).$$

In (ii) we have considered the case when B is an $m \times n$ matrix. Is the result (ii) true for the case when B is an $m \times p$ matrix? The possibility that the matrix $(AB)^*AB$ is singular when A^*A and B^*B are both non-singular does not exist. When A^*A and B^*B are both non-singular, so is $(AB)^*AB$. For

$$r(A) = m, \qquad r(B) = p,$$

while $r((AB)^*AB) = r(AB)$. But we have

$$r(AB) \geqslant r(A)+r(B)-m = p.$$

Since $(AB)^*AB$ is a $p \times p$ matrix, it is non-singular.

Corollary 1. *When AB is a square matrix*

$$c_{\min}(A^*A)c_{\min}(B^*B) \leqslant |c(AB)|^2 \leqslant c_{\max}(A^*A)c_{\max}(B^*B).$$

Proof. For a square matrix Q it is known that

$$c_{\min}(Q^*Q) \leqslant |c(Q)|^2 \leqslant c_{\max}(Q^*Q). \tag{2}$$

Take $Q = AB$. Then (2) together with (i) and (ii) of Theorem 1 gives the result. The above is a generalization of a result due to Nagy (**3**), who considers A and B as square matrices. His result was a generalization of one obtained by Roy (**4**) who required one of the (square) matrices A and B to be non-singular.

Corollary 2. *When A and B are normal (of the same order), then*

$$|c(A)|_{\min}|c(B)|_{\min} \leqslant |c(AB)| \leqslant |c(A)|_{\max}|c(B)|_{\max}.$$

Proof. When P is normal, we can find a unitary matrix U such that $P = U^*DU$, where D is a diagonal matrix having diagonal elements that are characteristic values of P. Now

$$P^*P = U^*D^*UU^*DU = U^*D^*DU.$$

This shows that the characteristic values of P^*P are equal to the square of the moduli of characteristic values of P when P is normal. This consideration together with Theorem 1 will give the result.

I shall obtain some more results about the characteristic values of the matrix AB. For a matrix $V = (v_{ij})$ of order $n \times p$ define

$$R_i(V) = \sum_{s=1}^{p} |v_{is}|, \qquad R(V) = \max_i R_i(V),$$

$$C_j(V) = \sum_{t=1}^{n} |v_{tj}|, \qquad C(V) = \max_j C_j(V).$$

Thus $R_i(V)$ is the sum of absolute values of elements in the ith row of V, and $C_j(V)$ is the sum of absolute values of the elements in the jth column of V. When V is *square*, it is known that

$$|c(V)| \leqslant \min\{R(V), C(V)\}. \tag{3}$$

Indeed, Barankin (**1**) has further improved this inequality and has shown that

$$|c(V)|^2 \leqslant \max_i\{R_i(V)C_i(V)\}. \tag{4}$$

Consider matrices $P = (p_{ij})$ of order $n \times m$ and $Q = (q_{ij})$ of order $m \times p$. If $V = PQ$, then V is of order $n \times p$. Now

$$\begin{aligned} R_i(V) = \sum_{s=1}^{p} |v_{is}| &= \sum_{s=1}^{p} \left| \sum_{k=1}^{m} p_{ik} q_{ks} \right| \\ &\leqslant \sum_s \sum_k |p_{ik}|\,|q_{ks}| = \sum_k \sum_s |p_{ik}|\,|q_{ks}| \\ &= \sum_k |p_{ik}| \sum_s |q_{ks}| = \sum_k R_k(Q)|p_{ik}| \\ &\leqslant R(Q)R_i(P) \leqslant R(P)R(Q). \end{aligned} \tag{5}$$

Thus $$R(V) \leqslant R(P)R(Q).$$

Also $$C_j(V) \leqslant C(P)C_j(Q), \qquad C(V) \leqslant C(P)C(Q). \tag{6}$$

In fact, $R(V)$ is a *modulus* of the matrix V and so satisfies the *norm* conditions [see (**5**)]. However, I have derived these results directly as I wish to use Barankin's theorem which requires the expressions $R_i(V)$, etc.

THEOREM 2. *When AB is square, then*

$$|c(AB)| \leqslant \min\{R(A)R(B), C(A)C(B)\}.$$

Proof. The result follows from (3), (5), (6). Theorem 2 is a generalization of some results obtained by Khan (**2**), who has considered square matrices A and B. My proof is direct and short. A weaker inequality can be deduced from Theorem 1. We know that

$$c(A^*A) \leqslant R(A^*A) = C(A^*A) \leqslant R(A)C(A)$$

and that $$c(B^*B) \leqslant R(B)C(B).$$

Thus $$|c(AB)|^2 \leqslant R(A)R(B)C(A)C(B).$$

The following theorem is sharper than Theorem 2.

THEOREM 3. *When AB is a square matrix,*

$$|c(AB)|^2 \leqslant C(A)R(B)\max_i\{R_i(A)C_i(B)\}.$$

Proof. Take $V = AB$. From (5) and (6) we have

$$R_i(AB) \leqslant R(B)R_i(A), \qquad C_i(AB) \leqslant C(A)C_i(B).$$

The result follows from (4). I have already obtained this theorem for the case when A and B are square matrices [to appear in the *American Math. Monthly*]. It is possible slightly to improve the above result. I state this in the corollary.

COROLLARY. *When AB is a square matrix, then*

$$|c(AB)|^2 \leqslant \min[C(A)R(B)\max_i\{R_i(A)C_i(B)\},$$
$$C(B)R(A)\max_j\{R_j(B)C_j(A)\}].$$

Proof. When AB is a square matrix, then so is BA. The non-zero characteristic values of AB and BA are the same. Hence

$$|c(AB)|_{\max} = |c(BA)|_{\max}.$$

By Theorem 3 we get

$$|c(AB)|^2 \leqslant C(B)R(A)\max_j\{R_j(B)C_j(A)\},$$

which gives the result.

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INTEGRAL TRANSFORMS OVER CERTAIN FUNCTION SPACES (II)

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1. Introduction

IN a previous paper (**2**), I discussed the characterization by means of integrals of all unitary transformations between a pair of function spaces of L^2 type, not necessarily the same. In the simplest case, the L^2 spaces are constructed as follows. Let (δ, s_0, μ) be a σ-finite measure space in which s_0 is the σ-ring of subsets of δ, and μ the non-negative measure. The function space, say $l^2(\mu)$, is then the class of measurable functions f, g,... with values in the field of complex numbers for which

$$\|f\| = \left\{\int\limits_{\delta} |f|^2\, d\mu\right\}^{\frac{1}{2}} < \infty, \qquad (f,g) = \int\limits_{\delta} f\bar{g}\, d\mu.$$

With the system (δ, s_0, μ), we associate a ring of sets $s \subset s_0$, with elements p, q,... each of finite measure μ, with the following property. If $p_1,..., p_n$ are disjoint sets of s, $cp_1,..., cp_n$ their characteristic functions, $a_1,..., a_n$ arbitrary complex numbers, then the class of all simple functions $\sum a_r cp_r$ is dense in $l^2(\mu)$. It will be convenient to denote such a system by the symbols

$$[\delta, s_0, s, \{p, q,...\}, \mu, l^2(\mu), \{f, g,...\}]. \tag{1.1}$$

Suppose next that we have a second system of this type

$$[\Delta, S_0, S, \{P, Q,...\}, \nu, L^2(\nu), \{F, G,...\}]. \tag{1.2}$$

Any unitary transformation from $l^2(\mu)$ to $L^2(\nu)$ may be given by integrals involving 'kernels' $\gamma(P) = \gamma(P, x)$ and $\Gamma(p) = \Gamma(p, y)$, where P, p vary over all sets of S, s respectively, and x, y over δ, Δ. The $\mu\nu S$-kernel $\gamma(P)$ is in $l^2(\mu)$ for every $P \in S$, is finitely additive over S, and is the zero element of $l^2(\mu)$ whenever $\nu(P) = 0$. The $\nu\mu s$-kernel $\Gamma(p)$ is in $L^2(\nu)$ for every $p \in s$, is finitely additive over s, and is the zero element of $L^2(\nu)$ whenever $\mu(p) = 0$.

In the present paper I consider two problems involving unitary transformations. In the first, we are given two systems of type (1.1) and two of type (1.2), with spaces $l^2(\mu_1)$, $l^2(\mu_2)$, $L^2(\nu_1)$, $L^2(\nu_2)$, and kernels defining unitary transformations T_1, T_2 between $l^2(\mu_1)$ and $L^2(\nu_1)$, and between $l^2(\mu_2)$ and $L^2(\nu_2)$. It is shown that the products of the kernels (in an

appropriate sense) define a unitary transformation between the Cartesian product spaces $l^2(\mu_1\times\mu_2)$, $L^2(\nu_1\times\nu_2)$. The second problem is to find expressions for the kernels of the repeated transformation $T_2(T_1)$ in terms of the kernels of the given unitary transformations T_1, T_2.

The basic results concerning a pair of systems (1.1), (1.2) will be required. These results, which constitute Theorems 1 and 2 of (**2**) may be summarized as follows.

To each unitary transformation T from $l^2(\mu)$ to $L^2(\nu)$ correspond a unique $\mu\nu S$-kernel γ and a unique $\nu\mu s$-kernel Γ satisfying for all $p, q \in s$, $P, Q \in S$ the relations

$$\int_p \bar{\gamma}(P)\,d\mu = \int_P \Gamma(p)\,d\nu, \tag{1.3}$$

$$\int_\delta \gamma(P)\bar{\gamma}(Q)\,d\mu = \nu(P\cap Q), \tag{1.4}$$

$$\int_\Delta \Gamma(p)\bar{\Gamma}(q)\,d\nu = \mu(p\cap q). \tag{1.5}$$

If $Tf = F$, $T^{-1}F = f$, then

$$\int_P F\,d\nu = \int_\delta f\bar{\gamma}(P)\,d\mu, \tag{1.6}$$

$$\int_p f\,d\mu = \int_\Delta F\bar{\Gamma}(p)\,d\nu, \tag{1.7}$$

again for all $p\in s$, $P\in S$.

Conversely, given a $\mu\nu S$-kernel γ and a $\nu\mu s$-kernel Γ for which (1.3), (1.4), (1.5) *hold, then* (1.6), (1.7) *define a unitary transformation $Tf = F$ from $l^2(\mu)$ to $L^2(\nu)$ and its inverse.*

2. Cartesian product of kernels

Let A be a ring over δ_1, B a ring over δ_2. Then the ring $A\times B$ in $\delta_1\times\delta_2$ is the set of all finite disjoint unions of rectangles $p\times q$, where $p\in A$, $q\in B$. The terminology is that of Halmos (**1**) Chapter VII, and particularly § 33.

Let Γ_1, Γ_2 be set functions which are defined and finitely additive over A, B respectively, and define a set function $\Gamma(\mathbf{p})$ over $A\times B$ by the relations

$$\mathbf{p} = \bigcup_i (p_i\times q_i), \qquad \Gamma(\mathbf{p}) = \sum_i \Gamma_1(p_i)\Gamma_2(q_i), \tag{2.1}$$

where $p_i\in A$, $q_i\in B$. It is known that this defines a set function which is finitely additive over $A\times B$.

For the sake of completeness, a sketch of the proof is included here.

Suppose first that $\mathbf{p}$ is itself a rectangle. If $p_1 \cap p_2 \neq 0$, then $q_1 \cap q_2 = 0$, and we may replace the rectangles $p_1 \times q_1$, $p_2 \times q_2$ by

$$(p_1 - p_2) \times q_1,\ (p_1 \cap p_2) \times (q_1 \cup q_2),\ (p_2 - p_1) \times q_2,$$

without altering $p = \bigcup p_i$, $q = \bigcup q_i$ or the value of $\sum \Gamma_1(p_i)\Gamma_2(q_i)$, empty sets being rejected. By applying this process systematically, we may replace the sets (p_i) by disjoint sets. In order to prove that, when $\mathbf{p} = p \times q$, $p_i \times q_i$ not empty,

$$p = \bigcup p_i, \qquad q = \bigcup q_i, \qquad \sum \Gamma_1(p_i)\Gamma_2(q_i) = \Gamma_1(p)\Gamma_2(q), \tag{2.2}$$

it will be sufficient, therefore, to consider the case in which (p_i) are disjoint.

Now, if $y \in q$, then $(x, y) \in p \times q$ for every $x \in p$, so there will be a point (x_i, y) in $p \times q$ for which $x_i \in p_i$, $y \in q_i$. It follows that $q_i = q$, and then (2.2) must hold.

Finally, let $\mathbf{p}$ be any element of $A \times B$,

$$\mathbf{p} = \bigcup_i (p_i \times q_i) = \bigcup_j (p'_j \times q'_j).$$

Then

$$p_i \times q_i = \bigcup_j \{(p_i \times q_i) \cap (p'_j \times q'_j)\} = \bigcup_j \{(p_i \cap p'_j) \times (q_i \cap q'_j)\}$$

is represented as a union of disjoint rectangles. By (2.2),

$$\sum_i \Gamma_1(p_i)\Gamma_2(q_i) = \sum_{i,j} \Gamma_1(p_i \cap p'_j)\Gamma_2(q_i \cap q'_j).$$

Thus the set function Γ is single-valued, and it is clearly finitely additive.

3. Cartesian product spaces

Suppose given four systems

$$[\delta_1, s_0^{(1)}, s_1, \mu_1, l^2(\mu_1)],\ [\delta_2, s_0^{(2)}, s_2, \mu_2, l^2(\mu_2)], \tag{3.1}$$

$$[\Delta_1, S_0^{(1)}, S_1, \nu_1, L^2(\nu_1)],\ [\Delta_2, S_0^{(2)}, S_2, \nu_2, L^2(\nu_2)], \tag{3.2}$$

as in (1.1), (1.2). The Cartesian product of the systems (3.1), denoted by $[\delta, \mathbf{s}_0, \mathbf{s}, \mu, l^2(\mu)]$, is defined as usual by the relations

$$\delta = \delta_1 \times \delta_2, \quad \mathbf{s}_0 = s_0^{(1)} \times s_0^{(2)}, \quad \mathbf{s} = s_1 \times s_2, \quad \mu = \mu_1 \times \mu_2.$$

It is assumed that the measure spaces in (3.1), (3.2) are measurable and σ-finite, so that $(\delta, \mathbf{s}_0, \mu)$ is σ-finite, and generates the l^2 space $l^2(\mu)$. Similarly we obtain from (3.2) the Cartesian product system

$$[\Delta, \mathbf{S}_0, \mathbf{S}, \nu, L^2(\nu)].$$

Clearly $\mathbf{s}$, $\mathbf{S}$ satisfy the conditions of § 1 laid down for s, S.

Let Γ_1, Γ_2 be $\nu_1\mu_1 s_1$, $\nu_2\mu_2 s_2$-kernels respectively, and, as before, define

$$\Gamma(\mathbf{p}) = \sum_i \Gamma_1(p_i)\Gamma_2(q_i) \tag{3.3}$$

over all finite disjoint unions $\mathbf{p} = \bigcup (p_i \times q_i)$ of rectangles in $\mathbf{s}$. Then Γ is finitely additive over $\mathbf{s}$, and it is easy to see that Γ is a $\nu\mu\mathbf{s}$-kernel. Also, if γ_1 is a $\mu_1\nu_1 S_1$-kernel and γ_2 a $\mu_2\nu_2 S_2$-kernel, then

$$\gamma(\mathbf{P}) = \sum_j \gamma_1(P_j)\gamma_2(Q_j), \tag{3.4}$$

over all finite disjoint unions of rectangles $\mathbf{P} = \bigcup (P_j \times Q_j)$ in $\mathbf{S}$, is a $\mu\nu\mathbf{S}$-kernel.

The final result is as follows.

THEOREM 1. *Let T_1 be a unitary transformation from $l^2(\mu_1)$ to $L^2(\nu_1)$ with kernels γ_1, Γ_1, and let T_2 be a unitary transformation from $l^2(\mu_2)$ to $L^2(\nu_2)$ with kernels γ_2, Γ_2. If Γ, γ are defined by* (3.3), (3.4), *then the relations*

$$\int_{\mathbf{P}} F\,d\nu = \int_{\delta} f\bar{\gamma}(\mathbf{P})\,d\mu \quad (\textit{all } \mathbf{P} \in \mathbf{S}), \tag{3.5}$$

$$\int_{\mathbf{p}} f\,d\mu = \int_{\Delta} F\overline{\Gamma}(\mathbf{p})\,d\nu \quad (\textit{all } \mathbf{p} \in \mathbf{s}), \tag{3.6}$$

define a unitary transformation T from $l^2(\mu)$ to $L^2(\nu)$, and its inverse.

It must be shown that (1.3), (1.4), (1.5) hold in the present case. Using successively (3.4), (3.3), we have

$$\begin{aligned}\int_{\mathbf{p}} \bar{\gamma}(\mathbf{P})\,d\mu &= \sum_i \int_{p_i \times q_i} \sum_j \bar{\gamma}_1(P_j)\bar{\gamma}_2(Q_j)\,d\mu \\ &= \sum_{i,j} \int_{p_i} \bar{\gamma}_1(P_j)\,d\mu_1 \int_{q_i} \bar{\gamma}_2(Q_j)\,d\mu_2 \\ &= \sum_{i,j} \int_{P_j} \Gamma_1(p_i)\,d\nu_1 \int_{Q_j} \Gamma_2(q_i)\,d\nu_2,\end{aligned}$$

since T_1, T_2 are unitary, and their kernels satisfy (1.3). This is equal to

$$\sum_j \int_{P_j \times Q_j} \sum_i \Gamma_1(p_i)\Gamma_2(q_i)\,d\nu = \int_{\mathbf{P}} \Gamma(\mathbf{p})\,d\nu,$$

and so the result corresponding to (1.3) holds. Similarly, (1.5) holds for Γ_1 and Γ_2; so, if

$$\mathbf{p} = \bigcup (p_i \times q_i), \qquad \mathbf{q} = \bigcup (p'_j \times q'_j)$$

are in $\mathbf{s}$, then

$$\begin{aligned}\int_{\Delta} \Gamma(\mathbf{p})\overline{\Gamma}(\mathbf{q})\, d\nu &= \sum_{i,j} \int_{\Delta_1} \Gamma_1(p_i)\overline{\Gamma}_1(p_j')\, d\nu_1 \int_{\Delta_2} \Gamma_2(q_i)\overline{\Gamma}_2(q_j')\, d\nu_2 \\ &= \sum_{i,j} \mu_1(p_i \cap p_j')\mu_2(q_i \cap q_j') \\ &= \sum_{i,j} \mu\{(p_i \cap p_j') \times (q_i \cap q_j')\} \\ &= \sum_{i,j} \mu\{(p_i \times q_i) \cap (p_j' \times q_j')\} \\ &= \sum_{i} \mu\{(p_i \times q_i) \cap \mathbf{q}\} \\ &= \mu(\mathbf{p} \cap \mathbf{q}).\end{aligned}$$

This is the result corresponding to (1.5), and the final condition can be verified in the same way. This completes the proof. Clearly the result extends immediately to Cartesian products of n spaces of type $l^2(\mu_i)$ and n spaces of type $L^2(\nu_i)$. This generalization is included in the discussion of the Euclidean case given in § 5.

4. Relations between T_1, T_2, and T

We suppose in this section that the conditions of Theorem 1 are satisfied, and distinguish functions which are elements of $l^2(\mu_1)$, $l^2(\mu_2)$, $l^2(\mu)$, $L^2(\nu_1)$, $L^2(\nu_2)$, $L^2(\nu)$ by assigning to them the arguments x, y, (x, y), ξ, η, (ξ, η), respectively. The following notation will be used. If a condition is to be satisfied (or a result holds) for every x in a space except possibly over a set of measure μ zero, we shall say that the condition is to be satisfied (or the result holds) 'for all $x \bmod \mu$'.

THEOREM 2. *Suppose that the conditions of Theorem* 1 *are satisfied, and let $f = f(x,y)$ be any element of $l^2(\mu)$. Then there exists a function $F_y(\xi)$ which is in $L^2(\nu_1)$ for all (fixed) $y \bmod \mu_2$ and in $l^2(\mu_2)$ for all (fixed) $\xi \bmod \nu_1$ and for which*

$$T_2\{T_1 f(x,y)\} = T_2\{F_y(\xi)\} = Tf.$$

Also, there exists a function $F_x^{(1)}(\eta)$ which is in $L^2(\nu_2)$ for all (fixed) $x \bmod \mu_1$ and in $l^2(\mu_1)$ for all (fixed) $\eta \bmod \nu_2$ and for which

$$T_1\{T_2 f(x,y)\} = T_1\{F_x^{(1)}(\eta)\} = Tf.$$

Given f, for all $y \bmod \mu_2$, f is in $l^2(\mu_1)$; so $F_y(\xi) = T_1 f(x,y)$ is in $L^2(\nu_1)$ for all fixed $y \bmod \mu_2$. The functions $f(x,y)$, $F_y(\xi)$ are related as in § 1, and, in particular, the Parseval relation for T_1 gives

$$\int_{\Delta_1} |F_y(\xi)|^2\, d\nu_1(\xi) = \int_{\delta_1} |f(x,y)|^2\, d\mu_1(x).$$

By integrating this result, we obtain

$$\int_{\delta_2} d\mu_2(y) \int_{\Delta_1} |F_y(\xi)|^2 \, d\nu_1(\xi) = \int_{\delta} |f|^2 \, d\mu. \tag{4.1}$$

It follows that $F_y(\xi)$ is in $l^2(\mu_2)$ for all $\xi \bmod \nu_1$. Let $T_2 F_y(\xi) = \Phi(\xi, \eta)$. We must show that $\Phi = Tf$.

By (1.6),
$$\int_{P} F_y(\xi) \, d\nu_1(\xi) = \int_{\delta_1} f(x, y) \bar{\gamma}_1(P, x) \, d\mu_1(x) \tag{4.2}$$

holds for every $P \in S_1$ and all $y \bmod \mu_2$, while

$$\int_{Q} \Phi(\xi, \eta) \, d\nu_2(\eta) = \int_{\delta_2} F_y(\xi) \bar{\gamma}_2(Q, y) \, d\mu_2(y) \tag{4.3}$$

holds for every $Q \in S_2$ and all $\xi \bmod \nu_1$. Hence, using first (4.3) and then (4.2), we obtain

$$\begin{aligned}
\int_{P} d\nu_1(\xi) \int_{Q} \Phi(\xi, \eta) \, d\nu_2(\eta) &= \int_{P} d\nu_1(\xi) \int_{\delta_2} F_y(\xi) \bar{\gamma}_2(Q, y) \, d\mu_2(y) \\
&= \int_{\delta_2} \bar{\gamma}_2(Q, y) \, d\mu_2(y) \int_{P} F_y(\xi) \, d\nu_1(\xi) \\
&= \int_{\delta_2} \bar{\gamma}_2(Q, y) \, d\mu_2(y) \int_{\delta_1} f(x, y) \bar{\gamma}_1(P, x) \, d\mu_1(x) \\
&= \int_{\delta} f(x, y) \bar{\gamma}_1(P, x) \bar{\gamma}_2(Q, y) \, d\mu_1(x) \, d\mu_2(y).
\end{aligned} \tag{4.4}$$

The inversion of integrals is justified as follows.

$$\begin{aligned}
&\left\{ \int_{\delta_2} |\gamma_2(Q, y)| \, d\mu_2(y) \int_{P} |F_y(\xi)| \, d\nu_1(\xi) \right\}^2 \\
&\qquad \leqslant \int_{\delta_2} |\gamma_2(Q, y)|^2 \, d\mu_2(y) \int_{\delta_2} \left\{ \int_{P} |F_y(\xi)| \, d\nu_1(\xi) \right\}^2 d\mu_2(y) \\
&\qquad \leqslant \nu_2(Q) \int_{\delta_2} \left\{ \nu_1(P) \int_{P} |F_y(\xi)|^2 \, d\nu_1(\xi) \right\} d\mu_2(y)
\end{aligned}$$

by (1.4), and this is finite by (4.1). So (4.4) has been proved, and the result now follows by comparing (4.4), (3.5). The other results stated are obtained by a similar argument with T_1, T_2 interchanged.

5. The Euclidean case

This case arises when all spaces concerned are Euclidean, and integrals are of Lebesgue–Stieltjes type. Now the results take on a simpler form, since we may use for the special rings (s, $\mathbf{s}$, etc.) the rings of all finite disjoint unions of half-open intervals, which differ only when the spaces

concerned are of different dimension. Another simplification is due to the possibility of referring our conditions to a fixed point in the space. So there will be different Euclidean cases of Theorem 1, but the differences will be of a trivial nature. (Theorem 4 of (**2**) is formulated for the half-line $x \geqslant 0$.)

The case which is written out in detail in this section has applications in the theory of linear differential equations. We suppose that δ_i, Δ_j are all $(-\infty, \infty)$, and that there are n spaces δ_i and n spaces Δ_j. Throughout the section, statements involving i, j are supposed to hold for all $1 \leqslant i, j \leqslant n$.

All intervals used are supposed half-open, closed either to the left or the right—say the former. Let μ_i, ν_j be non-negative Radon measures, generating Lebesgue–Stieltjes integrals, and the spaces $l^2(\mu_i)$, $L^2(\nu_j)$ over $(-\infty, \infty)$. It will be convenient to write

$$\mu_i(\alpha,\beta) = \begin{cases} \int\limits_0^{\min(\alpha,\beta)} d\mu_i & (\alpha, \beta \geqslant 0), \\ \int\limits_{\max(\alpha,\beta)}^{0} d\mu_i & (\alpha, \beta < 0), \\ 0 & (\text{otherwise}), \end{cases}$$

and $\nu_j(\alpha,\beta)$ is defined in the same way in terms of integrals with respect to ν_j. There is clearly no loss of generality in assuming that the origin has measure μ_i and ν_i zero, and then conditions will be referred to the origin as fixed point.

The kernels of the previous theory are replaced by a set of $2n$ functions $\gamma_i(\alpha, x)$, $\Gamma_j(\alpha, x)$ of two variables which are defined for all α. Each $\gamma_i(\alpha, x)$ is defined for all $x \bmod \mu_i$, and each $\Gamma_j(\alpha, x)$ is defined for all $x \bmod \nu_j$.

THEOREM 3. *Let* $\gamma_i(\alpha, x)$, $\Gamma_j(\alpha, x)$ *satisfy, in addition, the following conditions*:

(i) *for each (fixed)* α, $\gamma_i(\alpha, x) \in l^2(\mu_i)$, $\Gamma_j(\alpha, x) \in L^2(\nu_j)$;

(ii) *if* α, α' *are such that* $\int\limits_{\alpha}^{\alpha'} d\nu_i = 0$, *then* $\gamma_i(\alpha, x) = \gamma_i(\alpha', x)$ *for all* $x \bmod \mu_i$; *if* α, α' *are such that* $\int\limits_{\alpha}^{\alpha'} d\mu_i = 0$, *then* $\Gamma_j(\alpha, x) = \Gamma_j(\alpha', x)$ *for all* $x \bmod \nu_j$;

(iii) $\int\limits_0^{\alpha} \bar{\gamma}_i(\beta, x)\, d\mu_i(x) = \int\limits_0^{\beta} \Gamma_i(\alpha, y)\, d\nu_i(y)$ $\quad$ (*all* α, β);

(iv) $\int\limits_{-\infty}^{\infty} \gamma_i(\alpha, x)\bar{\gamma}_i(\alpha', x)\, d\mu_i(x) = \nu_i(\alpha, \alpha')$ (*all* α, α');

(v) $\int\limits_{-\infty}^{\infty} \Gamma_j(\alpha, x)\overline{\Gamma}_j(\alpha', x)\, d\nu_j(x) = \mu_j(\alpha, \alpha')$ (*all* α, α').

Then the relations

$$\int\limits_0^{\beta_1} \dots \int\limits_0^{\beta_n} F(\xi_1,\dots,\xi_n)\, d\nu_1(\xi_1)\dots d\nu_n(\xi_n)$$
$$= \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} f(x_1,\dots,x_n)\bar{\gamma}_1(\beta_1, x_1)\dots\bar{\gamma}_n(\beta_n, x_n)\, d\mu_1(x_1)\dots d\mu_n(x_n), \tag{5.1}$$

$$\int\limits_0^{\alpha_1} \dots \int\limits_0^{\alpha_n} f(x_1,\dots,x_n)\, d\mu_1(x_1)\dots d\mu_n(x_n)$$
$$= \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} F(\xi_1,\dots,\xi_n)\overline{\Gamma}_1(\alpha_1, \xi_1)\dots\overline{\Gamma}_n(\alpha_n, \xi_n)\, d\nu_1(\xi_1)\dots d\nu_n(\xi_n) \tag{5.2}$$

determine a unitary transformation $Tf = F$ *from* $l^2(\mu)$ *to* $L^2(\nu)$ *and its inverse. As before,* $\mu = \mu_1 \times \dots \times \mu_n$, $\nu = \nu_1 \times \dots \times \nu_n$.

Let s denote the ring of finite disjoint unions of half-open intervals in $(-\infty, \infty)$. Let $p \in s$, and let it be fixed by the end-points

$$\alpha_1 < \alpha_2 \leqslant \alpha_3 < \alpha_4 \leqslant \dots < \alpha_{2N}.$$

We write

$$\gamma_i(p, x) = \sum_{n=1}^{N} \{\gamma_i(\alpha_{2n}, x) - \gamma_i(\alpha_{2n-1}, x)\},$$

$$\Gamma_j(p, x) = \sum_{n=1}^{N} \{\Gamma_j(\alpha_{2n}, x) - \Gamma_j(\alpha_{2n-1}, x)\}.$$

By (i), (ii), $\gamma_i(p, x)$ is a $\mu_i \nu_i s$-kernel, and $\Gamma_i(p, x)$ a $\nu_i \mu_i s$-kernel. It will be shown that these kernels satisfy the conditions (1.3)–(1.5) of § 1; the results stated will then follow immediately from Theorem 1.

Let $q \in s$ be fixed by end-points $\beta_1 < \beta_2 \leqslant \beta_3 < \beta_4 \leqslant \dots < \beta_{2M}$. Then, using (iii), we have

$$\int\limits_p \bar{\gamma}_i(q, x)\, d\mu_i = \sum_{n=1}^{N} \sum_{m=1}^{M} \int\limits_{\alpha_{2n-1}}^{\alpha_{2n}} \{\bar{\gamma}_i(\beta_{2m}, x) - \bar{\gamma}_i(\beta_{2m-1}, x)\}\, d\mu_i$$
$$= \sum_{n=1}^{N} \sum_{m=1}^{M} \int\limits_{\beta_{2m-1}}^{\beta_{2m}} \{\Gamma_i(\alpha_{2n}, y) - \Gamma_i(\alpha_{2n-1}, y)\}\, d\nu_i$$
$$= \int \Gamma_i(p, y)\, d\nu_i.$$

The analogue of (1.4) follows from (iv). For

$$\int_{-\infty}^{\infty} \gamma_i(p,x)\bar{\gamma}_i(q,x)\,d\mu_i$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{M}\int_{-\infty}^{\infty} \{\gamma_i(\alpha_{2n},x)-\gamma_i(\alpha_{2n-1},x)\}\{\bar{\gamma}_i(\beta_{2m},x)-\bar{\gamma}_i(\beta_{2m-1},x)\}\,d\mu_i$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{M} \{\nu_i(\alpha_{2n},\beta_{2m})-\nu_i(\alpha_{2n},\beta_{2m-1})-\nu_i(\alpha_{2n-1},\beta_{2m})+\nu_i(\alpha_{2n-1},\beta_{2m-1})\}$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{M}\int_I d\nu_i,$$

where I is the intersection of $(\alpha_{2n-1},\alpha_{2n})$, $(\beta_{2m-1},\beta_{2m})$. So the integral on the left is equal to $\nu_i(p\cap q)$, as required. Similarly, the analogue of (1.5) follows by using (v) in place of (iv). This completes the proof of the theorem.

6. Fourier kernels

In many applications the kernels γ, Γ are absolutely continuous. In § 4 of (2), a *Fourier kernel* was defined to be a function $\phi(x,y)$ satisfying (in the notation of § 1)

$$\int_{p\times P} |\phi(x,y)|^2\,d\mu d\nu < \infty, \qquad \gamma(P,x) = \int_P \bar{\phi}(x,y)\,d\nu(y), \tag{6.1}$$

for every $p\in s$, $P\in S$. If γ is given by (6.1), then (1.3) and

$$\Gamma(p,y) = \int_p \phi(x,y)\,d\mu(x) \tag{6.2}$$

are equivalent, so that γ, Γ are necessarily adjoint [§ 4 of (2)].

In this section we note the consequences of including in the conditions of Theorem 3 the assumption that γ_i are given in terms of Fourier kernels.

THEOREM 4. *If*

$$\gamma_i(\alpha,x) = \int_0^{\alpha} \bar{\phi}_i(x,y)\,d\nu_i(y), \qquad \Gamma_j(\alpha,y) = \int_0^{\alpha} \phi_j(x,y)\,d\mu_j(x), \tag{6.3}$$

where

$$\iint_R |\phi_i(x,y)|^2\,d\mu_i(x)d\nu_i(y) < \infty \tag{6.4}$$

for every finite rectangle R, *then* γ_i, Γ_j *satisfy the conditions* (ii), (iii) *of Theorem* 3. *If the remaining conditions of Theorem* 3 *are satisfied, and*

(iv) $\int\limits_{-\infty}^{\infty} \gamma_i(\alpha, x)\bar{\gamma}_i(\alpha', x)\, d\mu_i(x) = \nu_i(\alpha, \alpha')$ (*all* α, α');

(v) $\int\limits_{-\infty}^{\infty} \Gamma_j(\alpha, x)\overline{\Gamma}_j(\alpha', x)\, d\nu_j(x) = \mu_j(\alpha, \alpha')$ (*all* α, α').

Then the relations

$$\int\limits_0^{\beta_1} \dots \int\limits_0^{\beta_n} F(\xi_1,\dots,\xi_n)\, d\nu_1(\xi_1)\dots d\nu_n(\xi_n)$$

$$= \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} f(x_1,\dots,x_n)\bar{\gamma}_1(\beta_1, x_1)\dots\bar{\gamma}_n(\beta_n, x_n)\, d\mu_1(x_1)\dots d\mu_n(x_n), \quad (5.1)$$

$$\int\limits_0^{\alpha_1} \dots \int\limits_0^{\alpha_n} f(x_1,\dots,x_n)\, d\mu_1(x_1)\dots d\mu_n(x_n)$$

$$= \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} F(\xi_1,\dots,\xi_n)\overline{\Gamma}_1(\alpha_1, \xi_1)\dots\overline{\Gamma}_n(\alpha_n, \xi_n)\, d\nu_1(\xi_1)\dots d\nu_n(\xi_n) \quad (5.2)$$

determine a unitary transformation $Tf = F$ *from* $l^2(\mu)$ *to* $L^2(\nu)$ *and its inverse. As before,* $\mu = \mu_1 \times \dots \times \mu_n$, $\nu = \nu_1 \times \dots \times \nu_n$.

Let s denote the ring of finite disjoint unions of half-open intervals in $(-\infty, \infty)$. Let $p \in s$, and let it be fixed by the end-points

$$\alpha_1 < \alpha_2 \leqslant \alpha_3 < \alpha_4 \leqslant \dots < \alpha_{2N}.$$

We write

$$\gamma_i(p, x) = \sum_{n=1}^{N} \{\gamma_i(\alpha_{2n}, x) - \gamma_i(\alpha_{2n-1}, x)\},$$

$$\Gamma_j(p, x) = \sum_{n=1}^{N} \{\Gamma_j(\alpha_{2n}, x) - \Gamma_j(\alpha_{2n-1}, x)\}.$$

By (i), (ii), $\gamma_i(p, x)$ is a $\mu_i \nu_i s$-kernel, and $\Gamma_i(p, x)$ a $\nu_i \mu_i s$-kernel. It will be shown that these kernels satisfy the conditions (1.3)–(1.5) of § 1; the results stated will then follow immediately from Theorem 1.

Let $q \in s$ be fixed by end-points $\beta_1 < \beta_2 \leqslant \beta_3 < \beta_4 \leqslant \dots < \beta_{2M}$. Then, using (iii), we have

$$\int\limits_p \bar{\gamma}_i(q, x)\, d\mu_i = \sum_{n=1}^{N} \sum_{m=1}^{M} \int\limits_{\alpha_{2n-1}}^{\alpha_{2n}} \{\bar{\gamma}_i(\beta_{2m}, x) - \bar{\gamma}_i(\beta_{2m-1}, x)\}\, d\mu_i$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{M} \int\limits_{\beta_{2m-1}}^{\beta_{2m}} \{\Gamma_i(\alpha_{2n}, y) - \Gamma_i(\alpha_{2n-1}, y)\}\, d\nu_i$$

$$= \int \Gamma_i(p, y)\, d\nu_i.$$

The analogue of (1.4) follows from (iv). For

$$\int_{-\infty}^{\infty} \gamma_i(p,x)\bar{\gamma}_i(q,x)\,d\mu_i$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{M}\int_{-\infty}^{\infty} \{\gamma_i(\alpha_{2n},x)-\gamma_i(\alpha_{2n-1},x)\}\{\bar{\gamma}_i(\beta_{2m},x)-\bar{\gamma}_i(\beta_{2m-1},x)\}\,d\mu_i$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{M} \{\nu_i(\alpha_{2n},\beta_{2m})-\nu_i(\alpha_{2n},\beta_{2m-1})-\nu_i(\alpha_{2n-1},\beta_{2m})+\nu_i(\alpha_{2n-1},\beta_{2m-1})\}$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{M}\int_{I} d\nu_i,$$

where I is the intersection of $(\alpha_{2n-1},\alpha_{2n})$, $(\beta_{2m-1},\beta_{2m})$. So the integral on the left is equal to $\nu_i(p\cap q)$, as required. Similarly, the analogue of (1.5) follows by using (v) in place of (iv). This completes the proof of the theorem.

6. Fourier kernels

In many applications the kernels γ, Γ are absolutely continuous. In § 4 of (**2**), a *Fourier kernel* was defined to be a function $\phi(x,y)$ satisfying (in the notation of § 1)

$$\int_{p\times P} |\phi(x,y)|^2\,d\mu d\nu < \infty, \qquad \gamma(P,x) = \int_{P} \bar{\phi}(x,y)\,d\nu(y), \tag{6.1}$$

for every $p\in s$, $P\in S$. If γ is given by (6.1), then (1.3) and

$$\Gamma(p,y) = \int_{p} \phi(x,y)\,d\mu(x) \tag{6.2}$$

are equivalent, so that γ, Γ are necessarily adjoint [§ 4 of (**2**)].

In this section we note the consequences of including in the conditions of Theorem 3 the assumption that γ_i are given in terms of Fourier kernels.

THEOREM 4. *If*

$$\gamma_i(\alpha,x) = \int_{0}^{\alpha} \bar{\phi}_i(x,y)\,d\nu_i(y), \qquad \Gamma_j(\alpha,y) = \int_{0}^{\alpha} \phi_j(x,y)\,d\mu_j(x), \tag{6.3}$$

where

$$\iint_{R} |\phi_i(x,y)|^2\,d\mu_i(x)d\nu_i(y) < \infty \tag{6.4}$$

for every finite rectangle R, then γ_i, Γ_j satisfy the conditions (ii), (iii) *of Theorem* 3. *If the remaining conditions of Theorem* 3 *are satisfied, and*

(R_N) is any ascending sequence of intervals converging as $N \to \infty$ to the complete n-dimensional space, then, as $N \to \infty$,

$$\int_{R_N} f(x_1,\ldots,x_n)\phi_1(x_1,y_1)\ldots\phi_n(x_n,y_n)\, d\mu_1(x_1)\ldots d\mu_n(x_n) \tag{6.5}$$

converges to $F = Tf$ in the norm of $L^2(\nu)$, and

$$\int_{R_N} F(\xi_1,\ldots,\xi_n)\phi_1(x_1,\xi_1)\ldots\phi_n(x_n,\xi_n)\, d\nu_1(\xi_1)\ldots d\nu_n(\xi_n) \tag{6.6}$$

converges to f in the norm of $l^2(\mu)$.

To verify condition (ii), we have, for $\alpha' > \alpha$,

$$|\gamma_i(\alpha',x)-\gamma_i(\alpha,x)| \leqslant \int_\alpha^{\alpha'} |\phi_i(x,y)|\, d\nu_i(y)$$

$$\leqslant \left\{\int_\alpha^{\alpha'} d\nu_i(y) \int_\alpha^{\alpha'} |\phi_i(x,y)|^2\, d\nu_i(y)\right\}^{\frac{1}{2}},$$

which vanishes since, by hypothesis, the first factor vanishes. By (6.3),

$$\int_0^\alpha \bar{\gamma}_i(\beta,x)\, d\mu_i(x) = \int_0^\alpha d\mu_i(x) \int_0^\beta \phi_i(x,y)\, d\nu_i(y)$$

$$= \int_0^\beta d\nu_i(y) \int_0^\alpha \phi_i(x,y)\, d\mu_i(x)$$

$$= \int_0^\beta \Gamma_i(\alpha,y)\, d\nu_i(y),$$

which is (iii), the inversion being justified by (6.4) and the Schwarz inequality. We can now verify the second condition of (ii) as before.

For the latter part of the theorem, it will be sufficient to prove the result for (6.5), since precisely similar arguments apply to (6.6). Let F_N denote the integral (6.5), and let

$$h_N = \begin{cases} f & ((x_1,\ldots,x_n) \in R_N), \\ 0 & (\text{otherwise}). \end{cases}$$

By (5.1),

$$\int_0^{\beta_1} \ldots \int_0^{\beta_n} F_N\, d\nu_1 \ldots d\nu_n$$

$$= \int_{R_N} f\, d\mu_1 \ldots d\mu_n \int_0^{\beta_1} \phi_1(x_1,y_1)\, d\nu_1(y_1) \ldots \int_0^{\beta_n} \phi_n(x_n,y_n)\, d\nu_n(y_n).$$

If we may invert the integral on the right, it will follow that $F_N = Th_N$. To justify this inversion, we have, e.g., when $\beta_1,..., \beta_n > 0$,

$$\int_{R_N} |f|\, d\mu_1...d\mu_n \int_0^{\beta_1} |\phi_1|\, d\nu_1 ... \int_0^{\beta_n} |\phi_n|\, d\nu_n$$

$$\leqslant \|f\| \left\{ \int_{R_N} \left[\int_0^{\beta_1} |\phi_1|\, d\nu_1 ... \int_0^{\beta_n} |\phi_n|\, d\nu_n \right]^2 d\mu_1...d\mu_n \right\}^{\frac{1}{2}}$$

$$\leqslant A\|f\| \left\{ \int_{R_N} d\mu_1...d\mu_n \int_0^{\beta_1} |\phi_1|^2\, d\nu_1 ... \int_0^{\beta_n} |\phi_n|^2\, d\nu_n \right\}^{\frac{1}{2}},$$

where A is constant, and this is finite, by (6.4), for given N. As the transformation T is norm-preserving,

$$\|Tf - F_N\| = \|Tf - Th_N\| = \|f - h_N\| \to 0$$

as $N \to \infty$. This completes the proof. There will, of course, be a more general result involving Fourier kernels, following from Theorem 1.

7. Successive transformations

We consider in this section another consequence of the theorem of § 1 for systems of type (1.1). Suppose given three such systems

$$[\delta_1, s_0^{(1)}, s_1, \{p\}, \mu_1, l^2(\mu_1), \{f, g,...\}],$$

$$[\delta_2, s_0^{(2)}, s_2, \{q\}, \mu_2, l^2(\mu_2), \{F, G,...\}],$$

$$[\delta_3, s_0^{(3)}, s_3, \{r\}, \mu_3, l^2(\mu_3), \{\mathbf{F}, \mathbf{G},...\}],$$

and unitary correspondences $f \to F \to \mathbf{F}$. Then the transformation $f \to \mathbf{F}$ will be unitary, and we can find an expression for each of its kernels.

THEOREM 5. *If T_1 (kernels γ_1, Γ_1), T_2 (kernels γ_2, Γ_2), are unitary transformations from $l^2(\mu_1)$ to $l^2(\mu_2)$ and from $l^2(\mu_2)$ to $l^2(\mu_3)$ respectively, and $T = T_2(T_1)$, then the kernels γ, Γ of T are given by*

$$\int_p \bar{\gamma}(r,x)\, d\mu_1(x) = \int_r \Gamma(p,w)\, d\mu_3(w) = \int_{\delta_2} \Gamma_1(p,y)\, \bar{\gamma}_2(r,y)\, d\mu_2(y), \quad (7.1)$$

for all $p \in s_1$, $r \in s_3$.

If $p \in s_1$, and cp is its characteristic function, then by (1.7),

$$T_1(cp) = \Gamma_1(p).$$

[In fact such relations are fundamental in the proof of the theorem: see § 3 of (2).] Hence, by (1.6) applied to T_2,

$$\int_r T_2(T_1\, cp)\, d\mu_3 = \int_{\delta_2} \Gamma_1(p,y) \bar{\gamma}_2(r,y)\, d\mu_2(y)$$

for every $p \in s_1$, $r \in s_3$. Now $T_2(T_1\, cp) = T(cp) = \Gamma(p)$, so, using (1.3), we have (7.1). By hypothesis, Γ_1, γ_2 are in $l^2(\mu_2)$, so the integral over δ_2 must exist.

If T_1, T_2, T are given by Fourier kernels ϕ_1, ϕ_2, ϕ respectively, then formally

$$\phi(x,w) = \int\limits_{\delta_2} \phi_1(x,y)\phi_2(y,w)\,d\mu_2(y).$$

This can hold only in special cases since the first condition of (6.1) refers to sets p, P of finite measure, and (6.4) to finite rectangles. By (6.1), (7.1), we have generally

$$\int\limits_{p} d\mu_1(x) \int\limits_{r} \phi(x,w)\,d\mu_3(w) = \int\limits_{\delta_2} d\mu_2(y) \int\limits_{p} \phi_1(x,y)\,d\mu_1(x) \int\limits_{r} \phi_2(y,w)\,d\mu_3(w),$$

so that (as usual) the integrated form is valid for all $p \in s_1$, $r \in s_3$.

[*Added July* 1957.] I am indebted to Professor A. C. Zaanen for the following improvement in the results of (**2**):

The results stated in (**2**) *are true if the auxiliary ring s is replaced by any subclass s satisfying the conditions* (i) *and* (iv) *of* § 2.

For clearly (i) of Lemma 1 still holds, and the appeal to Lemma 2 in the proof of Theorem 1 is unnecessary. As regards Theorem 2, suppose that $\sum a_i\, cp_i = 0$ for every value of x. Then

$$(\textstyle\sum a_i\,\Gamma(p_i),\, cP) = \sum a_i(\Gamma(p_i),\, cP) = \sum a_i(cp_i,\, \gamma(P))$$

by condition (A), and this vanishes by hypothesis. Hence

$$(\textstyle\sum a_i\,\Gamma(p_i),\, F) = 0$$

for a class $\{F\}$ dense in L^2, and accordingly $\sum a_i\,\Gamma(p_i) = 0$. It follows that T_0 as previously defined is linear and isometric, and the proof is completed as before.

It should also be emphasized that conditions (A) and (K_1) imply the conditions (iii) and (iv) of § 3 for the kernel γ, and similarly for the kernel Γ.

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HENSEL'S LEMMA

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1. Introduction

IN 1904, Hensel (**2**) proved that certain approximate factorizations of polynomials over the ring of p-adic integers could be refined to exact factorizations. Krull (**3**) extended this to polynomials over maximally complete valuation rings. Another line of generalization, due to Cohen (**1**) and Nagata (**4**), extends the result to polynomials over complete local rings.

The purpose of this paper is to provide a direct proof of Hensel's lemma for pseudo-complete valuation rings. Hitherto this case has been derived from Krull's result and Kaplansky's theorem on the equivalence of maximal completeness and pseudo-completeness. As the main application of Hensel's lemma in this context is to show that an algebraic extension of a pseudo-complete field admits one and only one prolongation of the valuation of the field, it seems worth while to avoid the extrinsic idea of maximality.

My proof of Hensel's lemma consists of an adaptation of Nagata's version, which I use as a basis for transfinite induction. Since it turns out to be more convenient to use pseudo-valuations in this connexion than valuations proper, the result which we obtain applies to a slightly wider class of rings than those mentioned above.

The author is very much indebted to Dr. Gravett for his advice and assistance in the preparation of this paper.

2. Preliminaries

2.1. *Pseudo-valuations.* Let Γ denote an ordered commutative group and let ∞ be a symbol such that, for each γ in Γ,

$$\gamma < \infty, \qquad \infty+\gamma = \gamma+\infty = \infty+\infty = \infty.$$

Let R be a commutative ring with identity.† A mapping v of R on $\{\Gamma, \infty\}$ is said to be a ***pseudo-valuation*** of R, and then R (more precisely $\{R, v, \Gamma\}$) is said to be a ***pseudo-valuation ring*** if

(i) $v(r) \geqslant 0$ for all r in R, $\quad v(0) = \infty$, $\quad v(1) = 0$;

† I shall make the convention that throughout this paper all rings are commutative and have an identity unless it is expressly stated otherwise.

(ii) $v(rs) \geqslant v(r)+v(s)$ for all r, s in R;

(iii) $v(r \pm s) \geqslant \min\{v(r), v(s)\}$ for all r, s in R.

2.2. *Polynomials.* Let R be a pseudo-valuation ring. Let p lie in $R[x]$. If $p \neq 0$, we shall denote the degree of p (in x) by ∂p and the leading coefficient of p by Λp. If $p = 0$, we make the convention that $\partial p = 0$, $\Lambda p = 0$.

We extend the pseudo-valuation v of R to a pseudo-valuation W of $R[x]$ by setting

$$W(p) = \min v(a_i) \quad (i = 0,\ldots, n), \quad \text{where } p = a_0 x^n + \ldots + a_n,$$

and $\quad W(0) = \infty$.

W, as a function on R, agrees with v. It is easy to see that, for p and q in $R[x]$,

$$W(pq) \geqslant W(p)+W(q), \qquad W(p \pm q) \geqslant \min\{W(p), W(q)\},$$

so that W is a pseudo-valuation of $R[x]$ consistent with v.

If, for p, q in R, $W(p-q) > 0$, we write $p \equiv q$. If, for γ in Γ,

$$W(p-q) \geqslant \gamma,$$

we write $p \equiv q(\gamma)$. Here '$\equiv$', and for each γ in Γ, '$\equiv (\gamma)$' are equivalence relations of $R[x]$.

2.3. *Completeness.* Let R be a pseudo-valuation ring. A subset

$$\mathscr{S} = \{k_\gamma \mid \gamma \in \Gamma'\}$$

indexed by a subset Γ' of $\{\Gamma, \infty\}$ is said to be a *compatible set* if, whenever $\gamma \leqslant \rho$ (both γ and ρ being in Γ'), $v(k_\gamma - k_\rho) \geqslant \gamma$.

If $\mathscr{S}$ is a compatible set, any element k of R for which $v(k_\gamma - k) \geqslant \gamma$ for all γ in Γ' is said to be a *limit* of the set.

A pseudo-valuation ring is said to be *complete* if it contains at least one limit for each of its compatible sets.

LEMMA 1. *Let R be a complete pseudo-valuation ring and let the pseudo-valuation be extended to $R[x]$ as in § 2.2. Let $\{f_\gamma\}$ be a compatible set of $R[x]$ with the further property that there exists an integer K such that $\partial f_\gamma \leqslant K$ for all γ in Γ'. Then f_γ has a limit in $R[x]$.*

Proof. Suppose that, for each γ in Γ',

$$f_\gamma = \sum_{i=0}^{K} r_{\gamma i} x^i,$$

where each $r_{\gamma i}$ lies in R. Then, for each $i = 0, 1,\ldots, K$, $\{r_{\gamma i} \mid \gamma \in \Gamma'\}$ is a compatible set of R with a limit r_i (say) in R. It is easy to see that $\sum_{i=0}^{K} r_i x^i$ is a limit of $\{f_\gamma\}$.

3. Hensel's lemma

Let R be a complete pseudo-valuation ring. Let f, g, h, l, m be elements of $R[x]$ and b, c be positive integers for which

$$\Lambda g = 1, \quad \partial g = b, \quad h \neq 0, \quad \partial h \leqslant c, \quad f \neq 0,$$
$$f \equiv gh, \quad \partial f = b+c, \quad lg+mh \equiv 1.$$

For each p, q in $R[x]$ and γ in $\{\Gamma, \infty\}$, the symbol $T(p,q,\gamma)$ denotes that

$$\Lambda p = 1, \quad \partial p = b, \quad q \neq 0, \quad \partial q \leqslant c, \quad p \equiv g,$$
$$q \equiv h, \quad W(f-pq) = \gamma,$$

and S denotes the set of all triples (p,q,γ) for which $T(p,q,\gamma)$. S is non-empty since $T(g,h,\nu)$ where $\nu = W(f-gh)$.

If (p,q,γ), (p',q',γ') are triples with p, q, p', q' in $R[x]$ and γ, γ' in $\{\Gamma, \infty\}$, then $(p',q',\gamma') \geqslant (p,q,\gamma)$ denotes that

$$\text{(i)} \quad \gamma' \geqslant \gamma; \qquad \text{(ii)} \quad p \equiv p'(\gamma); \qquad \text{(iii)} \quad q \equiv q'(\gamma).$$

Note that $(p,q,\gamma) \geqslant (p',q',\gamma')$ and $(p',q',\gamma') \geqslant (p,q,\gamma)$ together imply that $\gamma = \gamma'$. The binary relation '$\geqslant$', applied to the set S, is a pre-order (i.e. a reflexive and transitive binary relation on S).

3.1. *The main theorem*

THEOREM. *Let R be a complete pseudo-valuation ring; let f, g, h, l, m be elements of $R[x]$ and b, c positive integers for which*

$$\Lambda g = 1, \quad \partial g = b, \quad h \neq 0, \quad \partial h \leqslant c, \quad f \neq 0,$$
$$f \equiv gh, \quad \partial f = b+c, \quad lg+mh \equiv 1.$$

Then there exist polynomials G, H in $R[x]$ such that

$$\Lambda G = 1, \quad \partial G = b, \quad H \neq 0, \quad \partial H \leqslant c,$$
$$G \equiv g, \quad H \equiv h, \quad W(f-GH) = \infty.$$

Proof. It is enough to show that there exist G, H in $R[x]$ for which $T(G,H,\infty)$. This is an immediate consequence of the following lemmas together with Zorn's lemma for pre-ordered sets.

LEMMA 2. *The pre-order '$\leqslant$' of S is inductive.*

LEMMA 3. *If (g,h,γ) is a maximal element of S, then $\gamma = \infty$.*

These lemmas are proved below.

COROLLARY TO THE MAIN THEOREM (Hensel's lemma). *Let the pseudo-valuation ring R be such that, for each r in R, $v(r) = \infty$ implies that $r = 0$. Then the polynomials G, H have the properties*

$$g \equiv G, \quad h \equiv H, \quad \Lambda G = 1, \quad \partial G = b, \quad \partial H = c, \quad f = GH.$$

This result contains as particular cases the theorems of Hensel and Krull referred to in § 1.†

3.2. *Proof of Lemma* 2. We have to prove that, if $\{X_\alpha \mid \alpha \in A\}$ is a subset of S indexed by a non-empty set A such that, for each α, β in A, either $X_\alpha \leqslant X_\beta$ or $X_\beta \leqslant X_\alpha$, then there exists X in S such that, for each α in A, $X_\alpha \leqslant X$.

There is no loss of generality in assuming that, for each α, β in A, if

$$X_\alpha = (p_\alpha, q_\alpha, \gamma_\alpha), \qquad X_\beta = (p_\beta, q_\beta, \gamma_\beta), \qquad X_\alpha \neq X_\beta,$$

then $\gamma_\alpha \neq \gamma_\beta$. We may therefore suppose that A is a subset of $\{\Gamma, \infty\}$, Γ' say, and, for each γ in Γ', $X_\gamma = (p_\gamma, q_\gamma, \gamma)$.

We observe that $\{p_\gamma\}$ is a compatible set of polynomials and that $\partial p_\gamma = b$ for each γ. By Lemma 1 there exists a polynomial p such that $p \equiv p_\gamma(\gamma)$; further, we can take $\Lambda p = 1$. Similarly, there is a polynomial $q \equiv q_\gamma(\gamma)$ for each γ in Γ'. It can easily be verified that, for each γ in Γ', $W(f-pq) \geqslant \gamma$, and hence that $T(p,q,\nu)$ (for some ν). For each γ in Γ', $(p_\gamma, q_\gamma, \gamma) \leqslant (p, q, \nu)$. This completes the proof of Lemma 2.

3.3. *Proof of Lemma* 3. Suppose that (g_1, h_1, γ_1) lies in S, and that $\gamma_1 < \infty$. We shall prove that (g_1, h_1, γ_1) is not a maximal element of S.

Let $\delta_1 = W(lg+mh-1), \qquad \delta_2 = W(f-gh), \qquad \delta_3 = W(g_1-g),$

$$\delta_4 = W(h_1-h), \qquad \delta_5 = W(f-g_1 h_1), \qquad \epsilon = \min \delta_j \ (j = 1,\ldots, 5).$$

Since $\delta_j > 0$ $(j = 1, 2,\ldots, 5)$, $\epsilon > 0$. We prove a subsidiary lemma.

To each $i = 0, 1,\ldots, b+c$ *there correspond* u_i, v_i *in* $R[x]$ *such that*

$$\text{(i) } u_i h_1 + v_i g_1 \equiv x^i(\epsilon); \qquad \text{(ii) } \partial u_i < b; \qquad \text{(iii) } \partial v_i \leqslant c.$$

Proof. Since $lg+mh \equiv 1(\epsilon)$, we have

$$lgx^i + mhx^i \equiv x^i(\epsilon).$$

Because $\Lambda g = 1$, we can find q_i in $R[x]$ such that $mx^i = q_i g + u_i$, where $\partial u_i < \partial g$. Hence

$$[x^i l + q_i h] g + u_i h \equiv x^i(\epsilon).$$

In the bracketed term replace by zero all the coefficients which are of value not less than ϵ, and denote the resulting polynomial by v_i. Then

$$v_i g + u_i h \equiv x^i(\epsilon). \tag{1}$$

For any i, $\qquad \partial(x^i - u_i h) \leqslant b+c.$

† Note that in the proofs of Lemmas 2 and 3 we use only the commutativity of the quotient ring R/P, where $P = \{r \mid r \in R,\, v(r) > 0\}$, and not (in essence) the commutativity of R. With slight modification this corollary provides a generalization of the results of Cohen and Nagata also.

Since g is monic and no non-zero coefficient of $v_i \equiv 0(\epsilon)$,

$$\partial(v_i g) = \partial(x^i - u_i h).$$

This shows that $\partial v_i \leqslant c$. To complete the proof of this lemma, note that $g_1 \equiv g(\epsilon)$ and $h_1 \equiv h(\epsilon)$, so that (1) gives

$$v_i g_1 + u_i h_1 \equiv x^i(\epsilon).$$

Returning to the proof of Lemma 3, we put

$$g_2 = g_1 + \sum_{i=0}^{r+s} a_i u_i, \qquad h_2 = h_1 + \sum_{i=0}^{r+s} a_i v_i,$$

where

$$g_1 h_1 - f = \sum_{i=0}^{r+s} a_i x^i.$$

Clearly, $v(a_i) \geqslant \epsilon$ for $i = 0,..., b+c$. Let $W(f - g_2 h_2) = \gamma_2$. We now prove

(i) $(g_2, h_2, \gamma_2) \geqslant (g_1, h_1, \gamma_1)$, $\gamma_2 \geqslant \gamma_1 + \epsilon$; (ii) $T(g_2, h_2, \gamma_2)$.

For (i) we have $\quad g_2 - g_1 = \sum a_i u_i \equiv 0(\gamma_1)$

since $\gamma \leqslant v(a_i)$, and similarly $h_2 - h_1 \equiv 0(\gamma_1)$.

$$f - g_2 h_2 = -\sum a_i x^i + \sum a_i g_1 v_i + \sum a_i h_1 u_i + \sum\sum a_i a_j u_i v_j,$$

whence

$$\gamma_2 \geqslant \min_{i,j}\{W(a_i(g_1 v_i + h_1 u_i - x^i)), W(a_i a_j u_i v_j)\} \geqslant \gamma_1 + \epsilon.$$

For (ii) we have $\quad \Lambda g_1 = \Lambda g_2 = 1$

since $\partial(g_2 - g_1) < \partial g_2$; further $\partial g_2 = \partial g_1 = b$. If $h_2 = 0$, we may replace it by any element of R of value not less than γ_1 (e.g. by a_1) without affecting the argument: accordingly we assume $h_2 \neq 0$. Now

$$\partial h_2 = \partial(h_1 + a_i v_i) \leqslant c.$$

Also $g_2 \equiv g$ since $g_2 \equiv g_1$, $g_1 \equiv g$. Similarly $h_2 \equiv h$. This shows that $T(g_2, h_2, \gamma_2)$.

Now we have $\quad (g_2, h_2, \gamma_2) \geqslant (g_1, h_1, \gamma_1)$,

and, since $\gamma_2 > \gamma_1$, $\quad (g_1, h_1, \gamma_1) \not\geqslant (g_2, h_2, \gamma_2)$.

This shows that (g_1, h_1, γ_1) is not maximal, and completes the proof of Lemma 3.

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ON A CLASS OF INTEGRAL FUNCTIONS

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1. Introduction

LET $k(u)$ be a function of bounded variation, possibly complex, defined in the interval $(0,1)$. Consider the integral function

$$D(z) = 1-K(z), \tag{1}$$

where $K(z) = \int\limits_0^1 k(u)e^{zu}\,du$ and $z = x+iy$. We may suppose that

$$k(0) = k(0+), \qquad k(1) = k(1-). \tag{2}$$

We restrict $k(u)$ by the condition

$$k(1) \neq 0. \tag{3}$$

Our object is to prove

THEOREM 1. *Given a positive δ, there is a positive η $(= \eta(\delta))$ such that, if each zero of $D(z)$ is the centre of a disk of radius δ, then, for z outside the disks, $|D(z)| > \eta$.*

As an application of Theorem 1, we prove an expansion theorem for functions $f(t)$, of bounded variation in $(0,1)$. Let μ_ν $(\nu = 1, 2,...)$ be the zeros, supposed all simple, of $D(z)$. Then

$$D'(\mu_\nu) = -K'(\mu_\nu) \neq 0. \tag{4}$$

We choose the notation so that

$$0 \leqslant |\mu_1| \leqslant |\mu_2| \leqslant \dots.$$

Let
$$\phi_\nu(t) = e^{\mu_\nu t}\int\limits_{1-t}^1 k(u)e^{\mu_\nu u}\,du, \qquad \psi_\nu(t) = e^{-\mu_\nu t}.$$

Then the ϕ, ψ are biorthogonal in $(0,1)$, and

$$\int\limits_0^1 \phi_\nu\psi_\nu\,dt = K'(\mu_\nu).$$

For a function f in $L(0,1)$ we write

$$\alpha_\nu = \frac{1}{K'(\mu_\nu)}\int\limits_0^1 f(u)\phi_\nu(u)\,du.$$

We shall prove

THEOREM 2. *If $f(t)$ is of bounded variation in* $(0, 1)$, *then as* $n \to \infty$,

$$\sum_1^n \alpha_\nu e^{-\mu_\nu t} - \frac{1}{\pi}\int_0^1 f(v)\frac{\sin n(t-v)}{t-v}\,dv$$

converges to zero boundedly in any open interval $(\beta, 1)$, *where* $0 < \beta < 1$.

Substantially this result was proved by Herglotz (**1**) on the assumption that $k(u)$ is real. The Herglotz proof, which is based on the theory of residues, breaks down if this assumption is not made. [See (**1**), top of page 96, where the integral may be divergent if a is purely imaginary.] The difficulty could be overcome by using the information provided by Theorem 1. However, the method used here, though also based on the theory of residues, is different from that given by Herglotz.

It should be remarked that, in a previous paper (**2**), the expansion of a function $f \in L(0, 1)$ in a series of exponentials $e^{\lambda_\nu t}$ was considered, where the λ_ν are the zeros of $K(z)$, and where $k(0)k(1) \neq 0$. The two problems need independent treatment.

2. We have

$$zK(z) = k(1)e^z - k(0) - \int_0^1 e^{zu}\,dk(u). \tag{5}$$

Let

$$M = 2\Big(\limsup |k| + \int_0^1 |dk|\Big). \tag{6}$$

Then

$$|zK(z)| < \begin{cases} M & (x < 0), \\ Me^x & (x > 0). \end{cases} \tag{7}$$

Clearly,

$$\lim_{x\to\infty} e^{-x}\int_0^1 e^{zu}\,dk(u) = 0,$$

uniformly in y. Hence we can write

$$zK(z) = k(1)e^z\{1+\tau(z)\}, \tag{8}$$

where $\tau(z)$ is an integral function of z with the property

$$\tau(z) \to 0 \quad \text{uniformly as } x \to \infty. \tag{9}$$

It follows from (7) that there is only a finite number of zeros of $D(z)$ to the left of any line $x = c$. The large zeros of $D(z)$ can be denoted by z_m $(= x_m + iy_m)$ for large integral $|m|$, and $x_m \to \infty$ as $|m| \to \infty$. At a zero of $D(z)$ we have

$$\frac{e^z}{z}\{1+\tau(z)\} = \frac{1}{k(1)},$$

so that, by (9), the enumeration for large $|m|$ can be chosen so that

$$z_m - \log z_m = -\log k(1) + 2\pi mi + \epsilon_m, \tag{10}$$

where $\epsilon_m \to 0$ as $|m| \to \infty$. We may suppose that

$$-\pi < \arg k(1) \leqslant \pi. \tag{11}$$

It easily follows from (10) that, for $m > 0$,

$$|z_m| = (2m+\tfrac{1}{2})\pi - \arg k(1) + o(1), \qquad \arg z_m \to \tfrac{1}{2}\pi, \tag{12}$$

$$|z_{-m}| = (2m+\tfrac{1}{2})\pi + \arg k(1) + o(1), \qquad \arg z_{-m} \to -\tfrac{1}{2}\pi. \tag{13}$$

We shall say that an integral function $F(z)$ 'has the δ-η property in a region R' if, given a positive δ, there is a positive η $(= \eta(\delta))$ such that, if each zero of $F(z)$ in R is the centre of a disk of radius δ, then, for z in R and outside the disks, we have $|F(z)| > \eta$. If R is the sum of n regions R_i and $F(z)$ has the δ-η property in each R_i, with different functions $\eta(\delta)$, then $F(z)$ has the δ-η property in R. The assertion of Theorem 1 is that $D(z)$ has the δ-η property in the whole plane.

If Y is an assigned positive number, then

$$\lim_{x \to -\infty} D(z) = 1, \qquad \lim_{x \to \infty} |D(z)| = \infty \tag{14}$$

uniformly for $|y| \leqslant Y$. Then $D(z)$ has only a finite number of zeros in the strip S: $|y| \leqslant Y$. By (14), $D(z)$ has the δ-η property in S. We shall prove that it has the property in $y > Y$; and a similar proof will apply to $y < -Y$.

Write $$z = x+iy = re^{i\theta} \quad (0 < \theta < \pi).$$

By (10), for all sufficiently large c, there is a y_c such that the zeros of $D(z)$ in the region $y \geqslant y_c$ satisfy the inequality $|x - \log r| < c$. We may suppose that

$$y_c > \log y_c + c + 1, \tag{15}$$

$$\log y_c > 2c. \tag{16}$$

It will appear that for Y we may choose y_c for any sufficiently large c.

We divide the region $y \geqslant y_c$ into three regions by the curves

$$x = \log r \pm c.$$

We shall prove that, in the region on the left defined by

$$y \geqslant y_c, \qquad x \leqslant \log r - c, \tag{17}$$

we have $|D(z)| < \tfrac{1}{2}$; and that, in the region on the right defined by

$$y \geqslant y_c, \qquad x \geqslant \log r + c, \tag{18}$$

we have $|D(z)| > 1$. Thus, $D(z)$ has the δ-η property in these regions.

It will then remain to prove that $D(z)$ has the δ-η property in the region defined by

$$y \geqslant y_c, \qquad |x-\log r| < c.$$

This will be done in § 4.

3. We require three lemmas.

LEMMA 1. *The equation*

$$x = \log|x+iy_c|-c$$

has one solution x_c; *and*

$$\log y_c - c < x_c < 1+\log y_c - c.$$

Proof. Let
$$f(x) = x-\tfrac{1}{2}\log(x^2+y_c^2)+c.$$

Then
$$f'(x) = 1-\frac{x}{x^2+y_c^2} > 0.$$

Further $f(\log y_c - c) < 0$, while

$$f(\log y_c - c+1) = 1-\tfrac{1}{2}\log\left\{1+\left(\frac{\log y_c - c+1}{y_c}\right)^2\right\} > 0,$$

by (15). Hence the result.

LEMMA 2. *For* $x \leqslant x_c$, $|K(x+iy_c)| < \frac{1}{2}$.

Proof. Write

$$k(u) = k_1(u)+i\,k_2(u), \qquad \exp iy_c u = \cos y_c u + i \sin y_c u.$$

Then K can be expressed as the sum of four integrals, each of which is numerically less than $\frac{1}{8}$. Consider, for example,

$$\begin{aligned} I &= \int_0^1 k_1(u)e^{xu}\sin y_c u\, du \\ &= \int_0^\alpha k_1(u)\sin y_c u\, du + e^x \int_\alpha^1 k_1(u)\sin y_c u\, du \\ &= I_1+I_2 \end{aligned}$$

for some α satisfying $0 \leqslant \alpha \leqslant 1$. On integrating by parts, we find

$$|I_1| < \frac{1}{y_c}\left(|k_1(0)|+|k_1(\alpha)|+\int_0^\alpha |dk_1|\right) < \frac{M}{y_c}.$$

Similarly,
$$|I_2| < \frac{Me^x}{y_c}.$$

But
$$e^x \leqslant e^{x_c} \leqslant y_c e^{1-c},$$

by Lemma 1. Hence

$$|I| < M\left(\frac{1}{y_c}+e^{1-c}\right) < \tfrac{1}{8}$$

if c is sufficiently large.

LEMMA 3. *For $x = \log r - c$, $y \geqslant y_c$, we have $|K(z)| < \frac{1}{2}$.*

Proof. For such x,

$$x > \log y - c \geqslant \log y_c - c > c,$$

by (16). By (9), we may suppose that $|\tau(z)| < \frac{1}{2}$. By (8),

$$\begin{aligned}|K(z)| &< \tfrac{3}{2}|k(1)|e^{x}r^{-1}\\ &< \tfrac{3}{2}|k(1)|e^{-c}\\ &< \tfrac{1}{2}\end{aligned}$$

if c is sufficiently large.

It is now easy to prove that $|D(z)| > \frac{1}{2}$ in the region (17). It suffices to prove that $|K(z)| < \frac{1}{2}$. This is true on the boundary, by Lemmas 2 and 3. If ϕ is an assigned number between $\frac{1}{2}\pi$ and π, we can find a point z_0 such that the region (17) is contained in a sector of angle ϕ whose vertex is z_0. Write $z' = (z-z_0)e^{-i\lambda}$, where λ is the argument of the ray of symmetry of the sector. Then $K(z) = K_0(z')$ say, and $K_0(z')$, like $K(z)$, is an integral function of order one. By the Phragmén–Lindelöf principle, it follows that $|K_0(z')| < \frac{1}{2}$ in the transformed region. Hence the result. In the region (18),

$$\begin{aligned}|K(z)| &> \tfrac{1}{2}|k(1)|e^{x}r^{-1}\\ &> \tfrac{1}{2}|k(1)|e^{c}\\ &> 2\end{aligned}$$

if c is sufficiently large. Hence $|D(z)| > 1$.

4. The relation

$$w = u+iv = z-\log z \quad (0 < \arg z < \pi) \tag{19}$$

effects a biuniform conformal transformation of the half-plane $y > 0$ on to the half-plane $v > -\pi$ slit along the line $v = 0$, $u \geqslant 1$. We have

$$u = r\cos\theta - \log r, \qquad v = r\sin\theta - \theta.$$

The region

$$R\colon |x-\log r| \leqslant c, \qquad y \geqslant y_c,$$

is transformed into a region R' which contains the set $|u| \leqslant c$, $v \geqslant y_c$, and is contained in the set $|u| \leqslant c$, $v \geqslant y_c-\pi$. For z in R we have

$$x = r\cos\theta \geqslant \log r - c > \log v - c.$$

Hence $x \to \infty$ uniformly as $v \to \infty$. Writing $\eta(w) = \tau(z)$, we see that $\eta(w) \to 0$ uniformly as $v \to \infty$.

Write $$E(w) = 1-k(1)e^{w}.$$

Then $$D(z) = E(w)-k(1)e^{w}\eta(w) = E_1(w),$$

say. The function $E(w)$, of period $2\pi i$, has the δ-η property in R'. The zeros of $E_1(w)$ approach those of $E(w)$ for large v, and $E(w)-E_1(w) \to 0$ uniformly as $v \to \infty$. Hence $E_1(w)$ has the δ-η property in R'. Finally, by (19), a disk of radius δ and centre z_0 $(= x_0+iy_0)$ with large y_0 is transformed into a region of the w-plane which contains the disk of centre w_0 $(= z_0-\log z_0)$ and radius $\frac{1}{2}\delta$. Hence $D(z)$ has the δ-η property in R. This completes the proof of Theorem 1.

5. The contours C_n

We shall now construct the sequence of contours which will be used in the proof of Theorem 2. These contours are obtained from large circles centred at the origin on replacing not more than two arcs which are at less than unit distance from zeros of $D(z)$ by arcs of circles of unit radius centred at these zeros.

With the notation $\mu_1, \mu_2,\ldots$ of § 1 for the zeros of $D(z)$, we see from (12) and (13) that, for large n, the circle Γ_n: $|z| = |\mu_n|$ is at a distance greater than π from all the zeros μ_ν $(\nu \neq n)$ except possibly for one value n' (say) of ν. Moreover, $\operatorname{im}\mu_n$, $\operatorname{im}\mu_{n'}$ are large, and of opposite signs.

Let γ_ν denote the circle $|z-\mu_\nu| = 1$. Replace the arc of Γ_n which lies inside γ_n by the arc of γ_n which lies outside Γ_n, obtaining a contour Γ_n'. If $\gamma_{n'}$ does not meet Γ_n', then Γ_n' is the contour C_n. It has the property that it contains μ_ν for $\nu \leqslant n$ and does not contain μ_ν for $\nu > n$, and every μ_ν is at a distance not less than 1 from it. If $\gamma_{n'}$ meets Γ_n', we obtain the contour C_n by the following modification: if $n' > n$, we replace the arc of Γ_n' which is outside γ_n' by the arc of γ_n' which is inside Γ_n'; while, if $n' < n$, we replace the arc of Γ_n' which is inside γ_n' by the arc of γ_n' which is outside Γ_n'.

6. The proof of Theorem 2 will depend on the following lemma.

LEMMA 4. *There is a positive constant A such that, for z on C_n,*

$$\left|\frac{1}{D(z)}\right| < A, \qquad \left|\frac{e^z}{zD(z)}\right| < A.$$

Proof. Every point on C_n is at a distance not less than 1 from every zero of $D(z)$. By Theorem 1, there is an $\eta > 0$ such that $|D(z)| > \eta$ on

C_n. By (8) and (9), there is a $c > 0$ such that $|\tau(z)| < \frac{1}{2}$ for $x \geqslant c$. Let

$$A = \max\left(\frac{e^c}{\eta}, \frac{4}{|k(1)|}, \frac{4}{\eta|k(1)|}\right).$$

The first inequality of the lemma is satisfied. For the second inequality, we may suppose that $x > 0$. Write

$$F(z) = \frac{e^z}{zD(z)}.$$

If $x < c$, then $|F| < e^c/\eta$. If $x \geqslant c$, then

$$|K(z)| > \tfrac{1}{2}|k(1)|\, e^x r^{-1}. \tag{20}$$

If $|K(z)| > 2$, then

$$|D(z)| > |K(z)|-1 > \tfrac{1}{2}|K(z)| > \tfrac{1}{4}|k(1)|e^x r^{-1},$$

and $|F| < 4/|k(1)|$; while, if $|K(z)| \leqslant 2$, then by (20),

$$|F| < 4/\eta|k(1)|.$$

This completes the proof.

7. Proof of Theorem 2

Let t vary in the open interval $(\beta, 1)$, where $0 < \beta < 1$. Consider the expression

$$\begin{aligned} I_n &= \frac{1}{2\pi i}\int_{C_n} \frac{e^{-zt}}{D(z)}\, dz \int_0^1 f(v)e^{zv}\, dv \int_{1-v}^1 k(u)e^{zu}\, du \\ &= \frac{1}{2\pi i}\left(\int_{C_n^-} + \int_{C_n^+}\right) \\ &= I_1 + I_2, \end{aligned}$$

where C_n^-, C_n^+ denote the left and right halves of C_n, traversed in the positive direction. The value of I_n is

$$-\sum_1^n \alpha_\nu e^{-\mu_\nu t}.$$

We have
$$2\pi i I_1 = \int_{C_n^-} \frac{e^{z(1-t)}}{D(z)}\, dz \int_0^1 f(v)\, J\, dv,$$

where
$$J = \int_{1-v}^1 k(u)e^{z(u+v-1)}\, du.$$

On integrating by parts, we see that $zJ = O(1)$ since $\operatorname{re} z(u+v-1) \leqslant 0$. Hence, by Lemma 4,

$$I_1 = O\left(\int_{C_n^-} \left|e^{z(1-t)}\frac{dz}{z}\right|\right).$$

Hence I_1 is $O(1)$, and, for each t, is $o(1)$.

Next,

$$2\pi i I_2 = \int_{C_n^+} \frac{e^{-zt}}{D(z)}\,dz \int_0^1 f(v)e^{zv}\,dv\left\{1-D(z)-\int_0^{1-v} k(u)e^{zu}\,du\right\}$$

$$= A-B-C \text{ (say)}.$$

Here
$$A = \int_{C_n^+} \frac{e^{z(1-t)}}{zD(z)} A_1\,dz,$$

where
$$A_1 = z\int_0^1 f(v)e^{z(v-1)}\,dv.$$

On integrating by parts, we see that $A_1 = O(1)$. Further

$$B = \int_0^1 f(v)\,dv \int_{C_n^+} e^{z(v-t)}\,dz$$

$$= 2i\int_0^1 f(v)\frac{\sin r_n(v-t)}{v-t}\,dv,$$

where r_n is the radius of the large circle from which C_n is obtained by the modifications described in § 5. Finally

$$C = \int_{C_n^+} \frac{e^{z(1-t)}}{zD(z)}\,dz \int_0^1 f(v)\,C_1\,dv,$$

where
$$C_1 = z\int_0^{1-v} k(u)e^{z(u+v-1)}\,du.$$

On integrating by parts, we see that $C_1 = O(1)$. Thus

$$A+C = O\left\{\int_{C_n^+} \left|\frac{e^{z(1-t)}}{zD(z)}\,dz\right|\right\}.$$

We shall prove that
$$\int_{C_n^+} \left|\frac{e^{z(1-t)}}{zD(z)}\,dz\right| = o(1) \tag{21}$$

uniformly as $n \to \infty$. It will then follow that

$$\frac{1}{\pi}\int_0^1 f(v)\frac{\sin r_n(v-t)}{v-t}\,dv - \sum_1^n \alpha_\nu e^{-\mu_\nu t} \tag{22}$$

is $O(1)$, and, for each t, is $o(1)$. Since $f(v)$ is of bounded variation, we may replace r_n in (22) by n with an error which is $O(1)$, and which, for each t, is $o(1)$. This will complete the proof of Theorem 2.

To prove (21), we observe that the arcs (at most two) of unit circles which form part of C_n^+ have their centres $\xi+i\eta$ (say) at zeros of $D(z)$; and so $\xi \to \infty$ as $n \to \infty$. The contribution to the integral in (21) of these arcs is, by Lemma 4,

$$O(e^{-t(\xi-1)}) = o(1)$$

uniformly as $n \to \infty$. The remainder of C_n^+ is part of the semicircle

$$z = r_n e^{i\theta} \quad (|\theta| \leqslant \tfrac{1}{2}\pi).$$

By Lemma 4,
$$\left|\frac{e^z}{D(z)}\right| < A\min(r_n, e^x).$$

It suffices to prove that

$$\lim_{r\to\infty} \int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-tr\cos\theta}\min(r, e^x)\, d\theta = 0, \tag{23}$$

uniformly in t.

Let δ $(= \delta(r))$ be the number between 0 and $\frac{1}{2}\pi$ which satisfies

$$r = \exp(r\cos\delta).$$

Then
$$\tfrac{1}{2}\pi - \delta \sim r^{-1}\log r.$$

The integral in (23) is

$$2\int\limits_0^{\delta} e^{-tr\cos\theta} r\, d\theta + 2\int\limits_{\delta}^{\frac{1}{2}\pi} e^{(1-t)r\cos\theta}\, d\theta < 2\int\limits_{\frac{1}{2}\pi-\delta}^{\frac{1}{2}\pi} e^{-\beta r\sin\theta} r\, d\theta + 2(\tfrac{1}{2}\pi - \delta) r^{1-\beta}.$$

The second term is $O(r^{-\beta}\log r)$. The first term is

$$O\left(\int\limits_{\frac{1}{2}\pi-\delta}^{\frac{1}{2}\pi} e^{-2/\pi\,\beta r\theta} r\, d\theta\right) = O(e^{-2/\pi\,\beta r(\frac{1}{2}\pi-\delta)})$$
$$= O(r^{-2/\pi\,\beta}).$$

Hence the result.

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